A QUICK GLANCE OF SPLINE WAVELETS AND ITS APPLICATIONS

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Abstract

Polynomial spline wavelets have played a momentous role in the enlargement of wavelet theory. Due to their attractive properties compact support, good smoothness property, interpolation property, they are now provide powerful tools for many scientific and practical problems. As splines have specific formulae in both time and frequency domain, it greatly facilitates their manipulation. This paper is a summary of spline wavelet which started with splines and ends with the applications of spline wavelets. The paper is divided into four sections. The first section contains a brief introduction of splines and the second section is devoted to the discussion of spline wavelet construction via multiresolution analysis (MRA) with emphasis on B-spline wavelet. The underlying scaling functions are B-splines, which are shortest and most regular scaling function. In the third section, some remarkable properties of spline wavelets are discussed. The orthogonality and finite support properties make the spline wavelets useful for numerical applications and also have the best approximation properties among all the known wavelets. And the last section enclose a brief discussion of application of spline wavelets.

Keywords: *Splines, Multiresolution Analysis, Semi-Orthogonal Bases, Compact Support.*

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1. INTRODUCTION

Polynomial splines have a number of attractive properties that make them useful in variety of applications. These features include good smoothness properties, a simple analytical form and the fact that they have convenient representations in terms of simple basis functions. There is a great curiosity in the investigation of compactly supported wavelets. These interests are due to the computational capabilities of spline wavelets and the wide range of their applications. Thanks to some of their exceptional properties and mathematical simplicity, they are also applied and give very good result in various areas of applied sciences in comparison to other known wavelets.

The m^{th} order Cardinal B-spline with integer knot sequence $\mathbb{Z}$, defined recursively by

$$B_m(x) = (B_{m-1} * B_1)(x) = \int_0^1 B_{m-1}(x-t)dt \qquad (1)$$

where $B_1 = \chi_{[0,1]}$ is the characteristic function of the unit interval. The Cardinal B-spline has some very interesting properties which make them a perfect fit to use as scaling function in constructing wavelets. These characteristic properties are given in many books on splines. Here the properties are listed in form of a theorem as in [1].

Theorem 1: The m^{th} order cardinal B-spline B_m satisfies the following properties;

i) Supp $B_m = [0, m]$
ii) $B_m(x) > 0 \quad for\ 0 < x < m$
iii) $\sum_{k=-\infty}^{\infty} B_m(x-k) = 1 \quad \forall x$
iv) The Cardinal B-spline B_m and B_{m-1} are related by the identity $B_m(x) = \frac{x}{m-1} B_{m-1}(x) + \frac{m-x}{m-1} B_{m-1}(x-1)$
v) B_m is symmetric with respect to the centre of its support, namely $B_m\left(\frac{m}{2}+x\right) = B_m(\frac{m}{2}-x)$

Theorem 2: For any pair of integers m and j with $m \geq 2$, the family $S_j = \{2^{j/2} B_m(2^j x - k) : k \in \mathbb{Z}\}$ is a Riesz basis of subspace $V_j^m \in L^2(R)$ with Riesz bounds $A = \frac{1}{(2m-1)} \prod_{k=1}^{m-1} \frac{(1+\lambda_k)^2}{|\lambda_k|} > 0$ and B=1. And also $\sum_{k=-\infty}^{\infty} \left|\widehat{B_m}(\omega + 2\pi k)\right|^2 \geq \sum_{k=-\infty}^{\infty} \left|\widehat{B_m}(\pi + 2\pi k)\right|^2$

The proof of the above theorem are given in [1].

2. MULTIRESOLUTION ANALYSIS

Multiresolution analysis(MRA) is the basic tool for the construction of wavelet bases.

Definition:
A MRA generated by a function φ consist of sequence of closed subspaces $V_j,\ j \in \mathbb{Z}\ \ of\ L^2(R)$ satisfying

i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$
ii) $\overline{\bigcup_{j \in z} V_j} = L^2(R)$
iii) $\bigcap_{j \in Z} V_j = \{0\}$
iv) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$
v) There exist a function $\varphi \in V_0$ such that $\{\varphi(x - k : k \in \mathbb{Z}\}$ is an orthonormal basis.

The function φ defined in the last condition is called scaling function of MRA. For each subspace V_{j+1} there exist an orthonormal compliment W_j of V_{j+1} in V_j such that,

W_j is subspace of V_{j+1}

$$W_j \perp V_j$$

$$V_{j+1} = V_j \oplus W_j$$

And $W_k \perp W_l$ for all $k \neq l$

Hence under condition (i), (ii) (iii), it follows that $L_2 = \oplus_{k \in Z} W_k$. The spaces W_k , $k \in Z$ are called wavelet spaces of L_2 relative to the scaling function φ. A scaling function φ must be a function in $L_2(R)$ with $\int \varphi \neq 0$. Also since, $\varphi \in V_0$ is also in V_1 and $\{\varphi_{1,k} = 2^{j/2}(2x-k): k \in Z\}$ is a Riesz basis of V_1.

From section 1, it is clear that Cardinal B-spline of order m generates a MRA of $L_2(R)$ in the sense that $V_k^m = clos_{L^2}\{B_m(2^k.-j): j \in Z\}$. The two scale relation for cardinal B-spline of order m is written as

$$B_m(x) = \sum_{k=0}^{m} 2^{-m+1} \binom{m}{k} B_m(2x-k) \tag{2}$$

And Fourier transform of it is,

$$\widehat{B_m}(\omega) = P(z)\widehat{B_m}(\omega/2) \tag{3}$$

where $P(z) = \frac{1}{2}\sum_{k=0}^{m} 2^{-m+1} \binom{m}{k} z^k = (\frac{1+z}{2})^m$; $z = e^{-i\omega}/_2$

$P(z)$ is called the mask of the scaling function.

Theorem 3: A scaling function φ with refinement relation

$\varphi(x) = \sum_{-\infty}^{\infty} p_k \varphi(2x-k)$,forms an orthonormal basis[2] only if

$$|P(z)|^2 + |P(-z)|^2 = 1$$

In case of B-spline,

$$|P(z)|^2 + |P(-z)|^2 = cos^{2m}\frac{\omega}{4} + sin^{2m}\frac{\omega}{4} \leq cos^2\frac{\omega}{4} + sin^2\frac{\omega}{4} = 1.$$

Hence, B-spline form orthonormal basis for m=1 only. While in general, B_m generates semi-orthogonal wavelets with compact supports provided the two scale sequence of the scaling function B_m is given by

$$G(z) = \frac{E_{2m-1}(z)}{E_{2m-1}(z^2)}\overline{P(z)}, \ |z| = 1 \tag{4}$$

Where E_{2m-1} is the Euler-Forbenius polynomial of order $2m$-1 relative to B_m. Hence the two scale symbol of semi-orthogonal wavelet ψ with respect to B_m is given by

$$Q(z) = z^{-1}G(-z)K(z^2) = z^{-1}\frac{E_{2m-1}(z)}{E_{2m-1}(z^2)}\overline{P(-z)}K(z^2) \tag{5}$$

Where $K(z^2)$ is a Laurent series whose coefficient sequence is in l^1.

Thus the compactly supported semi-orthogonal wavelet ψ_m with minimum support that corresponding to the scaling function cardinal B-spline is given in term of its Fourier transform

$$\widehat{\psi_m}(\omega) = Q(z)\widehat{B_m}(\tfrac{\omega}{2}) \tag{6}$$

where $Q(z) = \frac{1}{2}\sum_n q_n z^n = \left(\frac{1-z}{2}\right)^m \sum_{k=0}^{2m-2} B_m(k+1)(-z)^k$

Hence the cardinal B-spline wavelet is

$$\psi_m(x) = \sum_n q_n B_m(2x-n) \tag{7}$$

Where $q_n = \frac{(-1)^n}{2^{m-1}}\sum_{l=0}^{m}\binom{m}{l} B_{2m}(n+1-l)$, $n = 0,\ldots,3m-2$

And has compact support $supp\, \psi_m = [0, 2m-1]$.

3. PROPERTIES

In [3], Unser briefly discussed the ten basic properties i,e time-frequency localization, best approximation property, two scale relation etc, holds by spline wavelet family. Here we discussed some more properties holds by B-spline wavelets.

3.1 Simple Manipulation

The cardinal B-spline defined as (n+1) fold convolution of characteristic function, thus have simple explicit form, which greatly simplifies its manipulation.

3.2 Generalized Linear Phase Filtering

Definition. 1) Let $f \in L^2(R)$, then f is said to have generalized linear phase if its Fourier transform satisfies

$$\hat{f}(\omega) = F(\omega)e^{-i(a\omega+b)} \quad , \quad a.e$$

Where $F(\omega)$ is a real-valued function and a, b are real constant.

2) A sequence $\{a_n\} \in l^1$ is said to have generalized linear phase if

$$A(e^{-i\omega}) = F(\omega)e^{-i(n\omega+b)}$$

For some real valued function $F(\omega)$, $n \in \frac{1}{2}Z$ and $b \in R$.

From these two definition it is very clear that $B_m, m \geq 2$ has generalized linear phase.

3.3 Spline Interpolation

The wavelet interpolation function $\bar{\psi}(x)$, is the wavelet function whose scaling function is the interpolation spline-

$$\bar{\psi}(x) = \sum d_i\, \bar{\varphi}(2x-i)$$

The Fourier transform of spline interpolation wavelet is

$$\hat{\hat{\psi}}(\omega) = \frac{c(-z)}{c(z^2)} Q(z) \hat{\hat{\varphi}}(\frac{\omega}{2})$$

where $\sum d_i \, z^i = \frac{c(-z)}{c(z^2)} Q(z)$

3.4 Oscillating Properties

The first order B-wavelet ψ_1 is simply the Haar function,

$$\psi_1 = \chi_{[0,1/2)} - \chi_{(1/2,1]}$$

Whose oscillating properties will be evident. Let,

$$f(x) = \sum_{j=0}^{N} c_j \, \psi_m(x - j)$$

Such that, $c_0 c_N \neq 0$, then the number of strong sign changes of f i,e $S^{-}(f)$ lies between N+3m-2 and 2N+3m-2. Setting N=0, $S^{-}(f) = 3m - 2$.

Thus ψ_m has 3m-2 zeros in the interior of its support [0,2m]. And hence, in contrast to the property of “total positivity” of cardinal B-spline B_m , the cardinal B-wavelet ψ_m possesses the property of “complete oscillation”.

4. APPLICATIONS

Due to the orthonormal bases, wavelets provide fast algorithms in numerical aspects in approximating because of their vanishing moment and small supports, which leads to sparse matrix. Semi-orthogonal compactly supported B-spline wavelets behave better and easier than other wavelets in a bounded interval. For these reason, they are good candidate for solving integral equations.

Splines are smooth and well behaved functions. Splines of degree n are (n-1) times continuously differentiable. As a result, splines have excellent approximation properties. B-splines and their wavelet counterparts have excellent localization properties so they are good templates for time-frequency signal analysis. The compactly supported B-spline wavelets have been found to be powerful tool in many scientific and practical application including mathematical approximation, the finite element method, image processing and compression and computer-aided geometric design.

5. CONCLUSION

Wavelets constructed via multiresolution analysis taking B-spline as scaling function generate orthonormal basis or semi-orthonormal basis depending on order of spline for the wavelet space. Constructed B-spline wavelets have a compact support and explicit formulae which reduces the calculation effort. These wavelets are easier to handle in bounded interval and due to polynomial function have excellent approximation property.

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