LATTICE POINTS ON THE HOMOGENEOUS CONE: $5(x^2+y^2)-9xy=23z^2$

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Abstract

Seven different method s of the non-zero non-negative solutions of homogeneous Diophantine equation $5(x^2 + y^2) - 9xy = 23z^2$ are obtained. Introducing the linear transformation $x = u + v$, $y = u - v$, $u \neq v \neq 0$ in $5(x^2+y^2) - 9xy = 23z^2$, it reduces to $u^2 + 19v^2 = 23z^2$. We are solved the above equation through various choices and are obtained seven different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: *The Method of Factorization, Integer Solutions, Linear Transformation, Relations and Special Numbers*

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Symbols used:

$t_{m,n}$ = Polynomial number of rank n with sides m
P_n^m = Pyramidal number of rank n with sides m
G_n = Gnomonic number of rank n
SO_n = Stella Octagular number of rank n

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1. INTRODUCTION

The number theory is king of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For a broad review of variety of problems, one may try to see [3-12]. For a broad review of variety of problems, one may try to see [3-12]. In 2014, Jayakumar**.** P, Meena. J [7] published a paper in finding the integer solutions of the homogeneous Diophantine equation $x^2 + 7y^2 = 16z^2$". In 2015, Jayakumar.P., Prabha.S [6] have published a paper in finding the integer solutions of the homogeneous Diophantine equation $x^2 + 9y^2 = 26z^2$". Inspired by these, we are observed in this work another interesting seven different non-zero integral solutions of the homogeneous Diophantine equation $5(x^2+y^2) - 9xy = 23z^2$ Further, some elegant properties among the special numbers and the solutions are observed.

2. DESCRIPTION OF METHOD

Consider the Diophantine equation

$$5(x^2 + y^2) - 9xy = 23z^2 \quad (1)$$

Take the linear transformations $x = u + v$,

$$y = u - v,\ u \neq v \neq 0. \quad (2)$$

Using (1) in (2), it gives us

$$u^2 + 19v^2 = 23z^2 \quad (3)$$

If we take $z = z(a, b) = a^2 + 19b^2 = (a + i\sqrt{19}\, b)$

$$(a - i\sqrt{19}\, b), \quad (4)$$

where a and b non-zero non- negative different integers, then we solve (1) through dissimilar method of solutions of (1) which are furnished below.

2.1 Method: I

We can write 23 as

$$23 = (2 + \sqrt{19}\, i)(2 - \sqrt{19}\, i) \quad (5)$$

Using (4) and (5) in (3) and applying the factorization process, this gives

$$(u + i\sqrt{19}\, v)(u - i\sqrt{19}\, v) = (2 + i\sqrt{19})(2 - i\sqrt{19})(a + i\sqrt{19}\, b)^2 (a - i\sqrt{19}\, b)^2$$

It gives us

$$(u + i\sqrt{19}\, v) = (2 + i\sqrt{19})(a + i\sqrt{19}\, b)^2 \quad (6)$$

$$(u - i\sqrt{19}\, v) = (2 - i\sqrt{19})(a - i\sqrt{19}\, b)^2 \quad (7)$$

Comparing both sides of (6) or (7), we obtain

$u = u(a, b) = 2a^2 - 38ab - 38b^2$
$v = v(a, b) = a^2 + 4ab - 19b^2$

In sight of (2), the solutions x, y are found to be

$$x = x(a, b) = 3a^2 - 34ab - 57b^2 \quad (8)$$

$y = y\,(a, b) = a^2 - 42ab - 19b^2$ (9)

Hence (4), (8) and (9) gives us two parametric the non-zero different integral values of (1).

Observations

1. $x\,(c, c+1) + y\,(c, c+1) + 76P_c + t_{4,c+1} = 4t_{4,c}$
2. $x\,(c, 2c^2-1) - 3y(c, 2c^2-1) = 92SO_c$
3. $x\,(c, c(c+1)) - 3y(c, c(c+1)) = 184Pc^5$
4. $x\,(c, d) - 3y\,(c, d) = G_{46cd} + 1$
5. $x\,(c, c) + y(c, c) + 148t_{4,c} = 0$
6. $x\,(a, a) + 88\ t_{4,a} = 0.$
7. $y\,(c, c) + 60\ t_{4,c} = 0$
8. $z\,(b, b) - 20\ t_{4,b} = 0.$
9. $y\,(1,d) + 19\ t_{4,d} + G_{2,d} = 0$
10. $x\,(c, c) - 3y(c, c) = 92\ t_{4,c}$
11. $x\,(c, 1) + y\,(c, 1) + G_{38c.} - 4t_{4,c} = -77$
12. $y\,(c, 1) + G_{21c} - t_{4,c} = -20$
13. $x\,(c, 1) + G_{17c} - 3t_{4,c} = -58$
14. $y\,(1, c) + z(1,c) + G_{21c} = 1$
15. $\frac{6}{5} z\,(1,1)$ = a nasty number

2.2 Method: II

We also write 23 as

$$23 = (-2 + i\sqrt{19})(-2 - i\sqrt{19}) \quad (10)$$

Using (4) and (10) in (3) and applying the factorization process, we obtain

$$(u + i\sqrt{19}\,v)(u - i\sqrt{19}\,v) = [(-2 + i\sqrt{19})(-2 - i\sqrt{19})$$
$$(a + i\sqrt{19}\,b)^2 (a - i\sqrt{19}\,b)^2$$

Comparing both sides of above, we are found to be the positive and negative factors as

$$(u + i\sqrt{19}\,v) = [(-2 + i\sqrt{19})(a + i\sqrt{19}\,b)^2 \quad (11)$$

$$(u - i\sqrt{19}\,v) = [(-2 - i\sqrt{19}\ \ (a - i\sqrt{19}\,b)^2] \quad (12)$$

Comparing both sides of (11) or (12), we obtain the real and imaginary parts as

$$u = u\,(a, b) = [-2a^2 - 38ab + 38b^2]$$

$$v = v\,(a, b) = [a^2 - 4ab - 19b^2]$$

In true of (2), the values of x, y are given by

$$x = x\,(a, b) = -a^2 - 42ab + 19b^2 \quad (13)$$

$$y = y\,(a, b) = -3a^2 - 34ab + 57b^2 \quad (14)$$

Hence (4), (13) and (14) gives us two parametric the non-zero different integral values of (1).

Observations

1. $x\,(c, 1) + z\,(c, 1) + G_{21c} = 37$
2. $x\,(1, b) - 19t_{4,b} + G_{21c} + 2 = 0$
3. $y\,(c, 1) + 3t_{4,c} + G_{17c} - 56 = 0$
4. $y\,(c, c(c+1)) - 3x(c, c(c+1)) - 184P_c^5$
5. $x\,(a, a) + y(a, a) + 4\ t_{4a} = 0$
6. $x\,(a, 2) + t_{4,a} + G_{42a} = 0$
7. $y\,(2, c) + G_{34c} - 57t_{4,c} = 0$
8. $y\,(c, c) + 24t_{4,c} = 0.$
9. $x\,(3,3c) - 57t_{4,c} + G_{189c} + 10 = 0$
10. $y\,(1,c) - 57t_{4,c} + G_{17c} + 4 = 0$

2.3 Method: III

Take 23 as $23 = \frac{(67 + i\sqrt{19})(67 - i\sqrt{19})}{196}$ (15)

For this choice, the corresponding values of x and y obtained from (2) are represented by

$$x = x\,(a, b) = \frac{1}{7}[34a^2 + 48ab - 646b^2] \quad (16)$$

$$y = y\,(a, b) = \frac{1}{7}[33a^2 - 86ab - 627b^2] \quad (17)$$

Since our intension is to find integer solutions, taking a as 7A and b as 7B in (4), (16) and (17), the related parametric integer values of (1) are found as

$$x = x\,(A, B) = 238A^2 + 336AB - 4522B^2 \quad (18)$$

$$y = y\,(A, B) = 231A^2 - 602AB - 4389 \quad (20)$$

Hence (18), (19) and (20) gives us two parametric the non-zero different integral values of (1).

Observations

1. $x\,(a, 2) - 238t_{4,a} - G_{336a} \equiv 0 \pmod 5$
2. $x\,(c., c) - z\,(c, c) + 4928t_{4,c} = 0$
3. $z\,(c, 1) - 49t_{4,}\,c \equiv 1 \pmod 2$
4. $z\,(a, a+1) + y(a, a+1) + 8911t_{4,a+1}266P_a - 469t_{4,a} = 0$
5. $y\,(c, d) + 4389t_{4.d} + G_{301cd} - 231t_{4,c} + 1 = 0.$
6. $x\,(a, a) + 3948t_{4,a} = 0$
7. $y\,(a, 1) - 231\ t_{4,a} + G_{301c} \equiv 0 \pmod 5$
8. $y\,(a, b) - z\,(a, b) + 53203\ t_{4,b} + G_{301ab} - 182t_{4,c} + 1 = 0$
9. $y\,(2, c) + G_{602c} + 4389t_{4,c} \equiv 1 \pmod 2$
10. $x\,(c, 1) - y(c, 1) + G_{133c} - 7t_{4,c} \equiv 0 \pmod 2$

2.4 Method: IV

Also take 23 as $23 = \frac{(-67 + i\sqrt{19})(-67 - i\sqrt{19})}{196}$ (21)

Following the analysis presented as in pattern- III and simplifying the corresponding non-zero different integer solutions of (1) are found as

$$x = x\,(A, B) = -231A^2 - 602AB + 4389B^2 \quad (22)$$

$$y = y\,(A, B) = -2348A^2 - 336AB + 4522B^2 \quad (23)$$

$z = z(a, b) = 49A^2 + 931B^2$ (24)

Hence (22), (23) and (24) gives us two parametric the non-zero different integral values of (1).

Observations

1. $y(1, a) + G_{168a} - 4522\, t_{4,a} \equiv 1 \pmod 2$
2. $x(1, a) + G_{301a} \equiv 0 \pmod 2$
3. $z(2,2b) - 3724 t_{4,b} \equiv 0 \pmod 2$
4. $x(a, a+1) - y(a, a+1) + 133\, t_{4,a+1} + 266 P_a - 7 t_{4,a} = 0$
5. $x(a, a) + y(a, a) - 7504 t_{4,a} = 0$
6. $y(b, b) - 3948\, t_{4,b} = 0$
7. $x(b, 1) + z(b,1) + 182\, t_{4,b} + G_{301,b} \equiv 1 \pmod 2$
8. $z(2, a) - y(2, a) + 4291\, t_{4,a} + G_{336a} \equiv 1 \pmod 2$
9. $x(b,1) - z(b,1) + 280 t_{4,b} + 280\, t_{4,b} + G_{301c} \equiv 1 \pmod 2$
10. $x(a+1, a+1) - 3556\, t_{4,a+1} = 0$

2.5 Method: V

Consider (3) as $u^2 - 4z^2 = 19(z^2 - v^2)$ and write it in the form of ratio as

$$\frac{u+2z}{z+u} = 19\frac{z-u}{u-2z} = \frac{A}{B},\ B \neq 0 \tag{25}$$

(25) is equivalent to the system of equations

$$Bu - 19v + (2B - 19A) z = 0 \tag{26}$$

$$-Au - Bv + (B+2A) z = 0 \tag{27}$$

By the cross multiplication method, the above equations yields as

$$u = -38 A^2 - 57AB - 2B^2 \tag{28}$$

$$v = 19A^2 - 4AB - B^2 \tag{29}$$

$$z = -19 A^2 - B^2 \tag{30}$$

In sight of (2), the solutions *x, y* are found to be

$$x = x(A, B) = -19A^2 - 61AB - 3B^2 \tag{31}$$

$$y = y(A, B) = -57A^2 - 53AB - B^2 \tag{32}$$

$$z = z(A, B) = -19 A^2 - B^2 \tag{33}$$

Thus (31), (32) and (34) gives us two parametric the non-zero different integral values of (1).

Observations

1. $x(1, 2a^2-1) - 3y(1, 2a^2-1) + 8SO_a - 152 t_{4,a} = 0$
2. $x(a, a(a+1)) - 3y(a, a(a+1)) + 16 P_c^5 - 152\, t_{4,c} = 0$
3. $x(a, 1) + y(a, 1) + 76\, t_{4,a} + G_{57c} \equiv 0 \pmod 5$
4. $x(a, a) - z(a, a) + 63\, t_{4,a} = 0$
5. $x(a, a) + y(a, a) + 194\, t_{4,a} = 0$
6. $y(a, 2) + 57\, t_{4,a} + G_{53a} \equiv 0 \pmod 5$
7. $x(a,2) + 19 t_{4,a} + G_{61a} + 13 = 0$
8. $z(2b, b) + 77 t_{4,b} = 0$
9. $3x(a, 1) - y(a, 1) + G_{65a} + 9 = 0$
10. $x(a, a+1) - z(a, a+1) + 38\, t_{4,a} + 4 t_{4,a+1} + 6\,1\, P_a = 0$

2.6 Method: VI

Consider (3) as $u^2 + 19v^2 = 23z^2 \times 1$ (34)

Write 1 as $1 = \frac{(9+i\sqrt{19})(9-i\sqrt{9})}{100}$ (35)

Using (34) and (35) in (3) and applying the factorization process, we obtain

$$(u + i\sqrt{19}\, v)(u - i\sqrt{19}\, v) = (2 + i\sqrt{19})(-2 - i\sqrt{19})$$

$$(a + i\sqrt{19}\, b)^2 (a - i\sqrt{19}\, b)^2 \frac{(9+i\sqrt{19})(9-i\sqrt{9})}{100}$$

Comparing both sides of above, we are found to be the positive and negative factors as

$$(u + i\sqrt{19}\, v) = [(2 + i\sqrt{19})(a + i\sqrt{19})^2 \frac{(9+i\sqrt{19})}{10}$$

$$(u - i\sqrt{19}\, v) = (2 - i\sqrt{19})\,(a - i\sqrt{19}\, b)^2 \frac{(9-i\sqrt{19})}{10}$$

Comparing both sides of (11) or (12), we obtain the real and imaginary parts as

$$u = \frac{1}{10}[-a^2 - 41ab + 19 b^2]$$

$$v = \frac{1}{10}[11a^2 - 2ab - 209 b^2]$$

In sight of (2), the solutions *x, y* are found to be

$$x = \frac{1}{5}[5a^2 - 210ab - 95b^2] \tag{36}$$

$$y = \frac{1}{5}[-6a^2 - 208ab + 114b^2] \tag{37}$$

Since our intension is to find integer solutions, taking a as 5A and b as 5B in (4), (36) and (37), the related parametric integer values of (1) are found as

$$x = x(A, B) = 25A^2 - 1050AB - 475B^2 \tag{38}$$

$$y = y(A, B) = -30A^2 - 1040AB + 570B^2 \tag{39}$$

$$z = z(A, B) = -25 A^2 + 475 B^2 \tag{40}$$

Thus (38), (39) and (40) gives us two parametric the non-zero different integral values of (1).

Observations

1. $x(2, a) + 475 t_{4,a} + G_{1050a} \equiv 0 \pmod 3$

2. $x(a, a+1) + y(a, a+1) + 5t_{4,a} + 2090P_a + 5t_{4,a+1} = 0$
3. $x(a, a(a+1)) + z(a, a(a+1)) + 2100 P_c^5 + G_{57c} - 50 t_{4,a} = 0$
4. $y(a, 1) + z(a, 1) + 5 t_{4,a} + G_{520a} \equiv 0 \pmod 2$
5. $x(a, a) + y(a, a) + 2000 t_{4,a} = 0$
6. $y(a, a+1) - 25 t_{4,a} - 475t_{4,a+1} = 0$
7. $y(a, a) - z(a, a) + 1000t_{4,a} = 0$
8. $y(2b, 2) + G_{2080b} + 120 t_{4,b} \equiv 1 \pmod 2$
9. $x(1, a) + z(a, 1) + G_{525a} \equiv 0 \pmod 7$
10. $x(a, 2a^2-1) + z(a, 2a^2-1) + 1050 SO_a - 50t_{4,a} = 0$

2.7 Method: VII

Also we write 1 as $1 = \frac{(5+i3\sqrt{19})(5-i3\sqrt{19})}{196}$ (41)

The following analysis presented as in pattern- VI and simplifying the corresponding non-zero different integer solutions of (1) are found as

$$x = x(A, B) = -126A^2 - 1792AB - 2394B^2 \quad (42)$$

$$y = y(A, B) = -203A^2 - 1134AB + 3857B^2 \quad (43)$$

$$z = z(A, B) = 49A^2 + 931B^2 \quad (44)$$

Observations:

1. $x(2, a) - 2394t_{4,a} + G_{1792a} \equiv 0 \pmod 5$
2. $z(a, 2) - 49t_{4,a} \equiv 0 \pmod 2$
3. $z(a, a+1) - x(a, a+1) + 1463t_{4,a+1} - 1792P_c - 175 t_{4,a} = 0$
4. $x(a, 1) + y(a, 1) + 329 t_{4,a} + G_{1463a} \equiv 0 \pmod 5$
5. $x(a, a) - y(a, a) + 2044 t_{4,a} = 0$
6. $y(a, a+1) - y(a, a+1) + 1463 t_{4,a+1} + 658P_c - 77t_{4,a} = 0$
7. $y(2, 2a) + G_{1134a} - 15428 t_{4,a} \equiv 1 \pmod 2$
8. $x(b, 1) + 126 t_{4,b} + G_{896a} \equiv 1 \pmod 2$
9. $z(1, a) + 931t_{4,a} \equiv 0 \pmod 7$
10. $y(1, a) - 3857t_{4,a} + G_{567a} \equiv 0 \pmod 2$

3. CONCLUSION

In this work, we observed various process of determining infinitely a lot of non-zero seven different integer values to the homogeneous Diophantine equation $5(x^2 + y^2) - 9xy = 23z^3$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

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