ON THE NON HOMOGENEOUS HEPTIC DIOPHANTINE EQUATION

\((x^2 - y^2)(8x^2 + 8y^2 - 14xy) = 19 (X^2 - Y^2) z^5\)

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Abstract

Five different methods of the non-zero non-negative solutions of non-homogeneous Heptic Diophantine equation \((x^2 - y^2)(8x^2 + 8y^2 - 14xy) = 19 (X^2 - Y^2) z^5\) are observed. Introducing the linear transformations \(x = u + v, y = u - v, X = 2u + v, Y = 2u - v, u \neq v \neq 0\) in \((x^2 - y^2) (8x^2 + 8y^2 - 14xy) = 19 (X^2 - Y^2) z^5\), it reduces to \(u^2 + 15v^2 = 19z^5\). We are solved the above equation through various choices and the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: The Diophantine Equation, Heptic Equation, Integral Solutions, Special Numbers, A Few Interesting Relation

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Notations used:

\(P_n^m\) : Pyramid number of rank \(n\) with size \(m\)

\(T_{n,m}\) : Polygonal number of rank \(n\) with size \(m\)

\(G_n\) : Gnomonic number of rank \(a\)

\(P_a\) : Pronic number of rank \(a\)

\(f_{13}^{r}, f_{53}^{r}\) : Fourth and fifth dimensional figurate number \(r\), whose generating polygon is a Triangle

\(f_{43}, f_{54}^{r}\) : Fourth and fifth dimensional figurate number \(r\), whose generating polygon is a Square

\(f_{45}^{r}, f_{55}^{r}\) : Fourth and fifth dimensional figurate number \(r\), whose generating polygon is a Pentagon

\(f_{46}^{r}, f_{56}^{r}\) : Fourth and fifth dimensional figurate number \(r\), whose generating polygon is a Hexagon

\(f_{47}^{r}, f_{57}^{r}\) : Fourth and fifth dimensional figurate number \(r\), whose generating polygon is a Heptagon

\(f_{48}^{r}, f_{58}^{r}\) : Fourth and fifth dimensional figurate number \(r\), whose generating polygon is a Octagon

1. INTRODUCTION

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1-21]. In 2014, Jayakumar.P, Sangeetha. K. [22] have published a paper in finding the integer solutions of the non-homogeneous Heptic Diophantine equation \((x^2 - y^2)(9x^2 + 9y^2 - 16xy) = 21 (X^2 - Y^2) z^5\). In 2015, the same authors Jayakumar.P, Sangeetha. K [23] published a paper in finding integer solutions of then non-homogeneous Heptic Diophantine equation \((x^2 - y^2)(5x^2 + 5y^2 - 8xy) = 13 (X^2 - Y^2) z^5\). Inspired by these, in this work, we are observed another interesting five different methods the non-zero integral solutions of the non-homogeneous Heptic Diophantine equation \((x^2 - y^2)(8x^2 + 8y^2 - 14xy) = 19 (X^2 - Y^2) z^5\)

2. DESCRIPTION OF METHOD

Consider the Heptic Diophantine equation

\((x^2 - y^2)(8x^2 + 8y^2 - 14xy)19(X^2 - Y^2)z^5\) \hspace{1cm} (1)

We introduce the linear transformation

\(x = u + v, y = u - v, X = 2u + v, Y = 2u - v\) \hspace{1cm} (2)

Using (2) in (1), gives to \(u^2 + 15v^2 = 19z^5\) \hspace{1cm} (3)

We solve (3) through various methods and hence obtain patterns of solutions to (1)
2.1 Method: I

Let us take \( z = z(a, b) = a^2 + 15b^2 \) \( (4) \)

where a and b are non - zero distinct integers

Take 19 as 19 = (2 + i√15) (2- i√15) \( (5) \)

Using (4) and (5) in (3) and applying the factorization process, define

\( u = u(a, b) = 2a^5 - 75a^4b - 300a^3b^2 + 2280a^2b^3 + 2450ab^4 - 3375b^5 \)

This gives us

\( v = v(a, b) = a^5 + 10a^4b - 150a^3b^2 - 300a^2b^3 + 1225ab^4 + 450b^5 \)

In sight of (2) the relating solutions of (1) are given by

\[ x = x(a, b) = 3a^5 - 65a^4b - 450a^3b^2 + 1950a^2b^3 + 3675ab^4 - 2925b^5 \]

\[ y = y(a, b) = a^5 - 85a^4b - 150a^3b^2 + 2530a^2b^3 + 1225ab^4 - 3825b^5 \]

\[ X = X(a, b) = 5a^5 - 140a^4b - 750a^3b^2 + 4200a^2b^3 + 6125ab^4 - 6300b^5 \]

\[ Y = Y(a, b) = 3a^5 - 160a^4b - 450a^3b^2 + 4800a^2b^3 + 2775ab^4 - 7200b^5 \]

\[ z = z(a, b) = a^2 + 15b^2 \]

Observations:

1. \( z(1, 1) \) is a perfect number

2. \( z(a, a) = 16 t_{a} = 0 \)

3. \( x(a, 1) - 3y(a, 1) - 190t_{a} + 3 + 5700 t_{a} \equiv 0 \) \( \text{mod} 2 \)

4. \( x(a, 0) - 3y(a, 0) = 0 \)

5. \( x(a, 1) - Y(a, 1) - 91t_{a} + 2850 t_{a} - G_{255} \equiv 0 \) \( \text{mod} 2 \)

6. \( x(a, 1) - Y(a, 1) = 0 \)

7. \( 3X(a, 1) - 5Y(a, 1) = 480 t_{a} + 1400 t_{a} - G_{2250} \equiv 0 \) \( \text{mod} 7 \)

8. \( 3X(a, 1) - 5Y(a, 1) = 0 \)

Each of the following is a nasty number

9. \( 2z(0, 1), \frac{x}{2}(1, 1), 2x(1, 0), 6x(1, 0), 2Y(1, 0) \)

2.2 Method: II

Also take (3) as \( u^2 + 15v^2 = 19z^5 \) \( = 1 \)

Take 1 as \( 1 = \left( \frac{1}{16} \right)(1 + i\sqrt{15})(1 - i\sqrt{15}) \)

The following techniques is same as Pattern-I, the relating non-zero dissimilar integral values of (1) are given by

\[ x = x(a, b) = \left\lfloor \frac{-10a^5 - 290a^4b - 1550a^3b^2 + 8700a^2b^3 - 24500ab^4 - 33750b^5}{4} \right\rfloor \]

\[ y = y(a, b) = \left\lfloor \frac{-16a^5 - 160a^4b + 2400a^3b^2 - 4800a^2b^3 - 19600ab^4 - 7200b^5}{4} \right\rfloor \]

\[ X = X(a, b) = \left\lfloor \frac{-23a^5 - 515a^4b + 2450a^3b^2 + 15450a^2b^3 - 28175ab^4 - 30375b^5}{4} \right\rfloor \]

\[ Y = Y(a, b) = \left\lfloor \frac{-26a^5 - 385a^4b + 4350a^3b^2 - 45525a^2b^3 - 35255ab^4 - 17325b^5}{4} \right\rfloor \]

Since our intention is to find integer solutions, taking a as 2a and b as 2b in (4), (8) \( (9), \left(10\right) \) and (14), the relating two parametric integer solutions of (1) are found as

\[ x = x(a, b) = \left\lfloor -80a^5 - 2320a^4b - 12400a^3b^2 + 5600a^2b^3 - 19600ab^4 - 20800b^5 \right\rfloor \]

\[ y = y(a, b) = \left\lfloor -128a^5 - 1280a^4b + 19200a^3b^2 - 38400a^2b^3 - 15680ab^4 - 57600b^5 \right\rfloor \]

\[ X = X(a, b) = \left\lfloor -184a^5 - 4120a^4b + 19600a^3b^2 + 123600a^2b^3 - 225400ab^4 - 243000b^5 \right\rfloor \]

\[ Y = Y(a, b) = \left\lfloor -208a^5 - 3080a^4b + 34800a^3b^2 + 108000a^2b^3 - 284200ab^4 - 138600b^5 \right\rfloor \]

Each of the following is a nasty number

\[ z = z(a, b) = 4a^2 + 60b^2 \]

Observations:

1. \( z(a, a) \) is a perfect square

2. \( x(a, 1) + 1920 f_{a}^{5} + 8832 f_{a}^{4} + 17767P_{a}^{5} - 14544P_{a} + G_{10535a} \equiv 0 \) \( \text{mod} 2 \)

3. \( y(a, 1) + 384 f_{a}^{5} - 2304 f_{a}^{4} - 44288P_{a}^{5} + 58752P_{a} + G_{49054a} \equiv 1 \) \( \text{mod} 2 \)

4. \( X(1,b) + 5832000 f_{b}^{5} + 863040 f_{b}^{4} + 2953600P_{b}^{5} + 2488960P_{b} - G_{4384220b} \equiv 1 \) \( \text{mod} 2 \)

5. \( Y(a, 1) = -6240 f_{a}^{5} + 21024 f_{a}^{4} - 8892P_{a}^{5} - 69348P_{a} + G_{126054a} \equiv 1 \) \( \text{mod} 2 \)

Each of the following is a nasty number

\[ \frac{3}{2} z(1, 0), \frac{1}{2} z(0, 1), \frac{3}{8} x(1, 0), \frac{3}{16} y(1, 0) \]
2.3: Method: III

In place of (5) take 19 as
\[ 19 = (13 + 3\sqrt{15})(13 + i3\sqrt{15}) \]  
(11)

The following technique is same as in Pattern-I, and applying a few calculations, the relating non-zero dissimilar integral solutions of (1) are given as
\[
\begin{align*}
x &= x(a, b) = 128a^2 - 1200a^3b + 19200a^3b^2 + 38400a^3b^3 + 156800ab^4 - 57600b^5 \\
y &= y(a, b) = 80a^4 - 2320a^4b - 12000a^3b^2 + 69600a^2b^3 + 69600ab^4 + 98000b^5 \\
X &= X(a, b) = 128a^2 - 3080a^4b + 27680a^3b^2 + 92400a^2b^3 + 204200ab^4 + 72400b^5 \\
Y &= Y(a, b) = 184a^3 - 4120a^4b - 34880a^3b^2 + 123600a^3b^3 + 225400ab^4 - 138600b^5 \\
z &= z(a, b) = 4a^2 + 60b^2
\end{align*}
\]

Observations:
1. \( z(1, 1) \) is a perfect square
2. \( z(a, a) - 64t_{4,a} = 0 \)
3. \( 10x(a, 1) - 16y(a, 1) - 24320 t_{4,a} + 107520 t_{a,a} - G_{72000} = 1 \) (mod 2)
4. \( 10x(a, 1) - 16y(a, 1) = 0 \)
5. \( y(a, 1) - 1920 f_{5,7}^a + 13440 f_{4,7}^a + 11040 P_a^3 - 78992P_a \equiv 1 \) (mod 2)
6. \( X(a, 1) = 6960 f_{5,6}^a + 18120 f_{4,8}^a - 644600 P_a^3 - 62670 P_a \equiv 1 \) (mod 2)
7. \( Y(a, 1) = 5520 f_{5,6}^a + 25292 f_{4,8}^a - 94672 P_a^3 - 82032 P_a \equiv 1 \) (mod 2)
Each of the following is a nasty number
8. \( 6z(1, 0), \frac{1}{2} z(0, 1), \frac{3}{16} x(1, 0), \frac{3}{10} y(1, 0), \frac{3}{29} X(1, 0) \)

2.4: Method: IV

Instead of (5) write 19 as
\[ 19 = \frac{1}{16}(17 + i3\sqrt{15})(17 - i3\sqrt{15}) \]  
(12)

The following technique is same as in Pattern-I, and doing some calculations, the relating non-zero dissimilar integral solutions of (1) are found as
\[
\begin{align*}
x &= x(a, b) = 144a^5 + 80a^4b - 2160a^3b^2 + 2400 a^2b^3 + 1476400ab^4 + 3600b^5 \\
y &= y(a, b) = 128a^5 + 1280a^4b - 19200a^3b^2 + 38400 a^2b^3 + 140800ab^4 - 57600b^5 \\
X &= X(a, b) = 280a^5 - 520a^4b - 42000a^3b^2 + 15600 a^2b^3 + 34300ab^4 - 23400b^5 \\
Y &= Y(a, b) = 264a^5 - 1880a^4b - 39600a^3b^2 + 56400 a^2b^3 + 323400ab^4 - 84600b^5 \\
z &= z(a, b) = 4a^2 + 60b^2
\end{align*}
\]

Observations:
1. \( z(2, 2) \) is a perfect square
2. \( z(a, a) - 64t_{4,a} = 0 \)
3. \( x(1, 0) \) is a perfect square
4. \( x(0, 1) \) is a perfect square
5. \( 8x(a, 1) - 9y(a, 1) - 1216 0 t_{4,a}^2 + 326400 t_{a,a} - G_{72000} = 1 \) (mod 2)
6. \( 8x(a, 0) - 9y(a, 0) = 0 \)
7. \( 33x(a, 1) - 35Y(a, 1) - 48640 t_{4,a}^2 + 1459200 t_{a,a} \equiv 0 \) (mod 5)
8. \( 33x(a, 1) - 35Y(a, 1) = 0 \)
Each of the following is a nasty number
9. \( \frac{1}{60} x(0, 1), \frac{3}{10} y(1, 0), \frac{1}{33} X(1, 0), \frac{1}{11} Y(1, 0) \)

2.3: Method: V

Instead of (5) write 1 as
\[ 1 = \frac{1}{64}(7 + i\sqrt{15})(7 - i\sqrt{15}) \]  
(13)

The following techniques is same as in Pattern-III, and doing some calculations, the relating non-zero dissimilar integral solutions of (1) are given by
\[
\begin{align*}
x &= x(a, b) = 32a^5 - 3080a^4b - 4800a^3b^2 + 6000a^2b^3 + 392000ab^4 - 122400b^5 \\
y &= y(a, b) = -40a^5 - 2720a^4b + 4800a^3b^2 + 4800a^2b^3 - 49000ab^4 - 120600b^5 \\
X &= X(a, b) = 28a^5 - 5380a^4b - 42000a^3b^2 + 114000a^2b^3 + 34300ab^4 - 243900b^5 \\
Y &= Y(a, b) = -144a^5 - 5420a^4b + 6600a^3b^2 + 102000a^2b^3 - 53900ab^4 - 242140b^5 \\
z &= z(a, b) = 4a^2 + 100b^2
\end{align*}
\]

Observations:
1. \( -y(1, 0) \) is a perfect square
2. \( z(a, a) - 104t_{4,a} = 0 \)
3. \( 5x(a, 1) + 4y(a, 1) + 26280 t_{4,a}^2 + 9600P_a^5 - 15600 t_{a,a} \equiv 0 \) (mod 5)
4. \( 5x(a, 1) + 4y(a, 1) = 0 \)
5. \( 7y(a, 1) + 10X(a, 1) + 72840 t_{4,a}^2 + 16800P_a^5 - 139200 t_{a,a} \equiv 0 \) (mod 5)
3. CONCLUSION

In this work, we have observed various process of determining infinitely a lot of non-negative different integer values to the non-homogeneous Heptic Diophantine equation $(x^2 - y^2)^2(8x^2 + 8y^2 - 14xy) = 19(X^2 - Y^2)z^2$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

4. REFERENCES


BIOGRAPHY

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