ON NON- HOMOGENEOUS BIQUADRATIC DIOPHANTINE EQUATION $5(x^2+y^2) -9xy = 23z^4$

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Abstract

Five different methods of the non-zero integral solutions of the homogeneous biquadratic Diophantine equation with five unknowns $5(x^2 + y^2) - 9xy = 23z^4$ are determined. Introducing the linear transformations $x = u + v$, $y = u - v$, $u \neq v \neq 0$ in $5(x^2 + y^2) - 9xy = 23z^4$, it reduces to $u^2 + 19v^2 = 23z^4$. We are solved the above equation through various choices and the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: *Quadratic, Non-Homogenous, Integer Solutions, Special Numbers, Polygonal, And Pyramidal Numbers*

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Notations used

$T_{m,n}$: Polygonal number of rank n with sides m.

p_n^m : Pyramidal number of rank m with side n

G_n: Gnomonic number of rank n

$f_{4,3}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Triangle

$f_{4,4}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Square

$f_{4,5}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Pentagon

$f_{4,6}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Hexagon

$f_{4,7}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Heptagon

$f_{4,8}^r$: Fourth dimensional figurate number of rank r, whose generating polygon is a Octagon.

1. INTRODUCTION

The number theory is the queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1-12]. In 2014, Jayakumar. P, Sangeetha. K, [12] have published a paper in finding the integer solutions of the homogeneous Biquaratic Diophantine equation $(x^3 - y^3)\ z = (W^2 - P^2)R^4$. In 2015, Jayakumar. P, Meena.J [14, 15] published two papers in finding integer solutions of the homogeneous Biquaratic Diophantine equation $(x^4 - y^4) = 26\ (z^2 - w^2)\ R^2$ and $(x^4 - y^4) = 40\ (z^2 - w^2)\ R^2$.Inspired by these, In this work, we are observed another interesting five different methods of the non-zero integral solutions of the non-homogeneous biquadratic Diophantine equation with three unknowns $5(x^2 + y^2) -9xy = 23z^4$. Further, some elegant properties among the special numbers and the solutions are observed.

2. DESCRIPTION OF METHOD

Consider the bi - quadratic Diophantine equation

$$5(x^2 + y^2) - 9xy = 23z^4 \quad (1)$$

We introduce the linear transformations

$$x = u + v,\ y = u - v,\ u \neq v \neq 0 \quad (2)$$

Using (2) in (1), it gives to $u^2 + 19v^2 = 23z^4$ (3)

We solved (3) through various choices and the different methods of solutions of (1) are obtained as follows.

2.1 Method: I

Consider (3) as $u^2 + 19v^2 = 19z^4 + 4z^4$ and write it as in the form of ratio

$$\frac{u + 2z^2}{19(z^2 + v)} = \frac{z^2 - v}{u - 2z^2} = \frac{a}{b},\ b \neq 0 \quad (4)$$

(4) is equivalent to the system of equations

$$6u - 199v + (2b - 19a)z^2 = 0 \quad (5)$$

$- au - bv + (b + 2a)z^2 = 0$ (6)

By the cross multiplication method, the above equations yields as

$$\left.\begin{array}{l} u = 38a^2 - 2b^2 + 38ab \\ v = -19a^2 + b^2 + 4ab \\ z^2 = 19a^2 + b^2 \end{array}\right\} \quad (7)$$

Putting $a = 2pq$, $b = 19p^2 - q^2$ in (7) and using (2), it gives us
$x = x(p, q) = -361p^4 - q^4 + 148p^2q^2 + 1596p^3q - 80pq^3$
$y = y(p, q) = -1083p^4 - 3q^4 + 338p^2q^2 + 1292p^3q - 64pq^3$
$z = 19p^2 + q^2$,
This gives us the non- zero different integer values to (1)

Observations:-

1. $x(1, p) + f^{p}_{4,6} + 154\, p_p^{5} - 227T_{4,p} - G_{798p} \equiv 0 \pmod 2$
2. $x(p, 1) + 8664\, f^{p}_{4,3} - 7524\, p_p^{5} - 357T_{4,p} - G_{1043p} = 0$
3. $y(1, p) + 72\, f^{p}_{4,8} - 15T_{4,p2} + 32\, p_p^{5} - 372T_{4,p} - G_{640p} = 0 \pmod 2$
4. $x(1, p) - y(1, p) - 48\, f^{p}_{4,5} + 4T_{4,p2} + 72\, p_p^{5} + 172T_{4,p} - G_{150p} \equiv 1 \pmod 2$
5. $\frac{6}{7} z(1, 4)$ is a Nasty number.

2.2 Method: II

In place of (4), let us take the form of ratio as

$$\frac{u + 2z^2}{z^2 - v} = \frac{19(z^2 + v)}{u - 2z^2} = \frac{a}{b}, b \neq 0 \quad (8)$$

The following techniques is similar as in the method - I,
The relating integer values to (1) are found as

$x = x(p, q) = 1143p^4 + 3q^4 - 342p^2q^2 + 1372p^3q - 68pq$
$y = y(p, q) = 381p^4 + q^4 - 114p^2q^2 + 1516p^3q - 84pq^3$
$z = 19p^2 + q^2$

Observations:-

1. $x(1, q) - y(1, q) - 24\, f^{q}_{4,4} - 16\, f_q^{5} + 31T_{4,q} + G_{74q} \equiv 0 \pmod 2$
2. $x(1, q) + y(1, q) - 96\, f^{q}_{4,7} + 16T_{4,q2} + 416\, p_q^{5} + 276T_{4,q} - G_{1448q} \equiv 1 \pmod 2$
3. $x(1, p) - 36\, f^{p}_{4,4} + 160\, p_p^{5} + 277T_{4,p} - G_{683p} \equiv 0 \pmod 2$
4. $y(1, p) - 6\, f^{p}_{4,6} + 174\, p_p^{5} + 29T_{4,p} - G_{758p} \equiv 0 \pmod 2$
5. $\frac{5}{7} z(1, 4)$ is a perfect Square.

2.3 Method: III

Take 23 as $23 = (2 + i\sqrt{19})(2 - i\sqrt{19})$ (9)

Write z as $z = z(a, b) = a^2 + 19b^2$ (10)

Using (9) and (10) is (3) and applying the factorization process, define
$(u + i\sqrt{19}\, v) = (2 + i\sqrt{19})(a + i\sqrt{19}\, b)^4$
This give us
$u = 2a^4 + 722b^4 - 228a^2b^2 - 4a^3b + 76ab^3$
$v = a^4 + 361b^4 - 114a^2b^2 + 8a^3b - 152ab^3$ (11)

Using (11) in (2), the relating integer values of (1) are furnished by
$x = x(a, b) = 3a^4 + 1083b^4 - 342a^2b^2 - 76ab^3 + 4a^3b$
$y = y(a, b) = a^4 + 361b^4 - 114a^2b^2 + 228ab + 12a^3b$
$z = z(a, b) = a^2 + 19b^2$

Observations:

1. $x(1, A) + y(1, A) - 8664\, f^{A}_{4,6} + 8360\, p_A^{5} - 5016T_{4,A} + G_{4A} \equiv 1 \pmod 2$
2. $x(1, A) - y(1, A) - 17328\, f^{A}_{4,8} + 3610T_{4,A2} + 32712\, p_A^{5} - 7296T_{4,A} - G_{1452A} \equiv 1 \pmod 2$
3. $x(1, A) - 25992\, f^{A}_{4,7} + 4332T^{4,A2} + 30476\, p_A^{5} - 7315T_{4,A} - G_{1085A} \equiv 0 \pmod 2$
4. $y(1, A) - 8664\, f^{A}_{4,5} + 722T^{4,A2} + 6764\, p_A^{5} - 19T_{4,A} + G_{367A} = 0$
5. $5z(1, 1)$ is a perfect Square.

2.4 Method: IV

In place of (9) take 23 as
$23 = \frac{(67 + i\sqrt{19})(67 - i\sqrt{19})}{196}$ (12)

The following techniques is same as in the method-III,
the relating integer values of (1) are found as

$x = x(A, B) = 18659A^4 + 67359712B^4 - 21271488a^2b^2 + 10010112ab^3 - 526848a^3b$

$y = y(A, B) = 181104A^4 + 5644408B^4 - 20563536a^2b^2 + 17934784ab^3 - 943936a^3b$

$z = z(A, B) = 196A^2 + 3724B^2$

Observations:

1. $x(1, A) - y(1, A) - 370291824\, f^{A}_{4,6} + 42618161\, p_A^{5} - 88952248T_{4,A} + G_{208544A} \equiv 1 \pmod 2$

2. x (1, A) +y (1, A) - 175209880 $f^{A}_{4,3}$ +860200096 p^{5}_{A} +414780296$T_{4,A}$+$G_{219747752A}$≡ 1 (Mod 2)

3. x (A, 1) - 2239104 $f^{A}_{4,4}$+ 2546432 p^{5}_{A} +20931232$T_{4,A}$ +$G_{5191648A}$≡ 1 (Mod 2)

4. y (A, 1)- 4346496 $f^{A}_{4,7}$ +724416$T_{4,A2}$+ 6958784 p^{5}_{A} + 18351872$T_{4,A}$– $G_{9148496A}$≡ 1 (Mod 2)

5. z (1, 0) is a perfect Square.

2.5 Method: V

Let us\take (3) as $u^2 + 19v^2 = 23z^4 * 1$ (13)

Take 1 as $1 = \frac{(9+i\sqrt{19})(9-i\sqrt{19})}{100}$ (14)

Using (9), (10) and (14) in (13) and applying the factorization process, define

$(u + i\sqrt{19}\, v) = (2+ i\sqrt{19})(a + i\sqrt{19}\, b)\ \frac{(9+i\sqrt{19})}{10}$

This gives us

$u = \frac{1}{10}[-a^4 - 361b^4 + 114a^2b^2 + 15884ab^3 - 836a^3b]$ (14)

$v = \frac{1}{10}[11a^4 + 3971b^4 - 1254a^2b^2 + 76ab^3 - 4a^3b]$ (15)

In sight of (2), the values of x, and y are

$x = \frac{1}{10}[10a^4 + 3610b^4 - 11a^2b^2 + 1664ab^3 - 84a^3b]$

$x = [a^4 + 3610b^4 - 114a^2b^2 + 1596ab^3 - 840a^3b]$ (16)

$y = \frac{1}{10}[-12a^4 - 4332b^4 + 239a^2b^2 + 1512ab^3 - 832a^3b]$

$y = \frac{1}{5}[-6a^4 - 2166b^4 + 684a^2b^2 + 756ab^3 - 416a^3b]$ (17)

As our intension is to find integer solutions, taking a as 5A and b as 5B in (4), (16) and (17), the relating parametric integer values of (1) are found as

$x = x(A, B) = 625A^4 + 225625B^4 - 71250A^2B^2 + 997500AB^3 - 52500A^3B$

$y = y(A, B) = -750A^4 - 270750B^4 + 85500A^2B^2 + 94500AB^3$ $52000A^3B$

$z = z(A, B) = 25A^2 + 475B^2$

Observations:

1. z (A, A) -500$T_{4,A}$ = 0

2. z (A, 0) – 25 $T_{4,A}$ = 0

3. z (0, B)- 475 $T_{4,B}$ = 0

4. $\frac{1}{5}$ z (1,1) is a perfect square

5. 6 x (A, 1) + 5 y (A, 1) + 1150000 p^{5}_{A} - 57500$T_{4,A}$ - $G_{3228750A}$+ 1 = 0

6. 6 x (A, 1) + 5 y (A, 1) = 0

7. x (1, 0) is a perfect square

8. x (A, 1) -300 $f^{A}_{4,7}$ + 108410 p^{5}_{A} - 17875$T_{4,A}$ – $G_{498875A}$≡ 0(Mod 2)

Each of the following is a nasty number

9. $\frac{6}{5}$ z (1, 0), $\frac{3}{50}$z (1,1), $\frac{6}{125}$x (1, 0), -$\frac{1}{25}$ y (1, 0)

3. CONCLUSION

In this work, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous bi-quadratic Diophantine equation $5(x^2 + y^2) - 9xy = 23z^4$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

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BIOGRAPHY

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