ON HOMOGENEOUS BIQUADRATIC DIOPHANTINEEQUATION: $x^4 - y^4 = 17\,(z^2 - w^2)\,R^2$

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Abstract

Five different methods of the non-zero non-negative solutions of non- homogeneous cubic Diophantine equation $x^4 - y^4 = 17(z^2 - w^2)R^2$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.

Keywords: *The Method of Factorization, Integer Solutions, Linear Transformation, Relations and Special Numbers*

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2010 Mathematics Subject Classification: 11D25

Symbols used:

$t_{m,n} = \frac{1}{2} n[(m-2)-(m-4)]$

$p_n^m = \frac{1}{6}[3n^2 + n^3(m-2) - n\,(m-5)]$

$G_n = 2n-1$

$Ct_{16,n} = 8n(n+1)+1$

$OH_n = 1/3n(2n^2+1)$

$SO_n = n(2n^2-1)$

$ky_n = (2n+1)^2 - 2carl_n$- carol number

1. INTRODUCTION

The Mathematics is the Queen of all sciences. In particular, the Number theory is the King of Mathematics. The Number theory, in particular Diophantine equations have a blend of interesting problems. Many greatest Mathematicians was fascinated by problems in Diophantine equations.

For a vide review, one may try to see [1-12]. In this work, we are observed a lot of infinitely the non –zero integer values of the cubic Diophantine equation. Some interesting relations among the special numbers and the solutions are found.

2. DESCRIPTION OF METHOD

Let us consider the cubic Diophantine equation

$$x^4 - y^4 = 17\,(z^2 - w^2)\,R^2 \qquad (1)$$

Consider the transformations

$x = u+v,\ y = u - v,\ z = 2uv+1,\ w = 2uv - 1$ (2)

Using (2) in (1), it gives us the equation

$u^2 + v^2 = 17R^2$ (3)

2.1 Method: I

We can write 17 as17 = (4+i) (4 –i) (4)

and $R = a^2 + b^2 = (a+ib)(a-ib)$ (5)

Using (4) and (5) in (3) and applying the process of factorization , it takes form as

$(u+iv)(u-iv) = (4+i)(4-i)(a+ib)^2(a-ib)$

The above equations give us

$(u+iv) = (4+i)(a+ib)^2$

$(u+iv) = (4-i)(a-ib)$

Comparing both sides of above equations, we obtain

$u = u(a,b) = 4a^2 - 4b^2 - 2ab$

$v = v(a,b) = a^2 - b^2 + 8ab$

Using the values of u and v in (2) we get the non-zero integer values of x, y, z and w and R of (1) are furnished by

$x = x(a,b) = 5a^2 - 5b^2 + 6ab$

$y = y(a,b) = 3a^2 - 3b^2 - 10ab$

$z = z(a,b) = 2(4a^4 + 4b^4 - 24a^2b^2 + 30a^3b - 30ab^3) + 1$

$w = w(a,b) = 2(4a^4 + 4b^4 - 24a^2b^2 + 30a^3b - 30ab^3) - 1$

$R = R(a,b) = a^2 + b^2$

Observations:

1.$3x[a(a+1),1] - 5y[a(a+1),1] - 68P_a = 0$

2.$3x[a(a-1),1] - 5y[a(a-1),1] - 68\,t_{4,a} \equiv 0 \pmod 2$

3.$z(b,1) - w(b,1) \equiv 0 \pmod 2$

4.$R(a+1, a+1) - 2t_{4,a} - G_{2a} \equiv 0 \pmod 3$

5. $x(2, a) + y(2, a) + 8P_a \equiv 0 \pmod 2$
6. $x(2,2) + y(2,2) + T_{16,2} = 0$
7. $[A(2A^2-1),1] - 5(SO_A)^2 - 6SO_A + 5T_{5,2} = 0$
8. $6x(1,0)$ *is a nasty number.*
9. $R(2,3) - S_2 = 0$
10. $3x[2a^2+1,a] - 5y[2a^2+1,a] - 204\,OH_a = 0$

2.2 Method: II

We can also take17 as 17 = (1 +4i) (1 – 4i) (6)

Using (5) and (6) in equation (3) it takes form as

$(u + iv)(u - iv) = (1+4i)(1-4i)(a+ib)^2(a-ib)^2$

It gives as

$(u + iv) = (1+4i)(a+ib)^2$

$(u - iv) = (1-4i)(a-ib)^2$

This leads to

$u = u(a, b) = a^2 - b^2 - 8ab$

$v = v(a, b) = 4a^2 - 4b^2 + 2ab$

Putting the values of u and v in (2), the non-zero different values of x, y, z,w and R of (1) are found as

$x = x(a, b) = 5a^2 - 5b^2 - 6ab$

$y = y(a, b) = -3a^2 + 3b^2 - 10ab$

$z = z(a, b) = 2(4a^4 + 4b^4 - 24a^2b^2 - 30a^3b + 30ab^3) + 1$

$w = w(a, b) = 2(4a^4 + 4b^4 - 24a^2b^2 - 30a^3b + 30ab^3) - 1$

$R = R(a, b) = a^2 + b^2$

Observations:

1. $3x[(2a-1)^2,1] + 5y[(2a-1)^2,1] + 68(G_a)^2 = 0$
2. $y(1,1) + R(1,1) + OH_2 \equiv 0 \pmod 2$
3. $3x[a, 2a^2-1] + 5y[2a^2-1] + 68So_a = 0$
4. $z(a,1) - W(a,1) - KY_1 \equiv 0 \pmod 5$
5. $z(1,a) - W(1,a) - Carl_2 \equiv 1 \pmod 2$
6. $z(1,a) - W(1,a) - Carl_2 - TK_1 = 0$
7. $x(a,a) - y(a,a) + 16T_{4,a} = 0$
8. $x(3,3) + y(3,3) + R(3,3) + T_{10,6} = 0$
9. $y(1,2a+1) - Ct_{16,a} - Sa + 10P_a - 8T_{4,a} = s^3$, where s is an integer
10. $x(b,b+1) + 5G_{2,b} + 6P_b$ *is an even integer.*
11. $z(2,2) + w(2,2) \equiv 0 \pmod 2$

2.3 Method: III

(3) can also be written as $1 * u^2 = 17R^2 - v^2$ (7)

Take as $u = 17a^2 - b^2 = (\sqrt{17}a + b)(\sqrt{17}a - b)$ (8)

Take 1 as $1 = (\sqrt{17} + 4)(\sqrt{17} - 4)$ (9)

Using (8) and (9) in (7), it takes the form as

$(\sqrt{17} + 4)(\sqrt{17} - 4)(\sqrt{17}a+b)^2(\sqrt{17}a-b)^2 = (\sqrt{17}R + v)(\sqrt{17}R - v)$

This gives us

$(\sqrt{17}+4)(\sqrt{17}a+b)^2 = (\sqrt{17}R+v)$

$(\sqrt{17}-4)(\sqrt{17}a-b)^2 = (\sqrt{17}R-v)$ (10)

This found as

$R = R(a, b) = 17a^2 + b^2 + 8ab$

$v = v(a, b) = 68a^2 + 4b^2 + 34ab$

Putting the values of u and v in (2), the non – zero different values of x, y, z, R and w of (1) are determined as

$x = x(a, b) = 85a^2 + 3b^2 + 34ab$

$y = y(a, b) = -51a^2 - 5b^2 - 34ab$

$z = z(a, b) = 2(1156a^4 - 4b^4 - 34ab^3 + 578a^3b) + 1$

$w = w(a, b) = 2(1156a^4 - 4b^4 - 34ab^3 + 578a^3b) - 1$

$R = R(a, b) = 17a^2 + b^2 + 8ab$

Observations:

1. $R(3a,3a) - 234t_{4,a} = 0$
2. $5x(a,1) - 3y(a,1) - Ct_{16,a} - Sa - 564t_{4,a} + G_{35a} \equiv 0 \pmod 3$
3. $R(a,2a-1) - Sa - Ct_{16,a} - 23t_{4,a} + G_{3n} \equiv 0 \pmod 2$
4. $3Z(1,1) - 3W(1,1) - OH_2 = 0$
5. $x[a,a(2a^2-1)] + y[a,a(2a^2-1)] - 34t_{4,a} + 2SO_a = 0$
6. $y(1,1) - P_8 = 0.$
7. $5x[A,(A+1)(2A+1)] - 3y[A,(A+1)(2A+1)] - 578T_{4,A} = 0.$
8. $x(1,A) + y(1,A) + 2T_{4,A} - T_{7,4} = 0.$
9. $R(2,2) \equiv 0 \pmod 2.$

2.4 Pattern: IV

Again take (3) as $1 * v^2 = 17R^2 - u^2$ (11)

Take 1 as $1 = \frac{(\sqrt{17}+1)(\sqrt{17}-1)}{16}$ (12)

Put $v = 17a^2 - b^2 = (\sqrt{17}a - b)(\sqrt{17}a + b)$ (13)

Using (12) and (13) in (11), it takes form as

$\frac{(\sqrt{17}+1)(\sqrt{17}-1)}{16}(\sqrt{17}a - b)^2(\sqrt{17}a+b)^2 = (\sqrt{17}R - u)(\sqrt{17}R+u)$ (14)

This gives us

$$\left.\begin{aligned} R &= R(a, b) = \tfrac{1}{4}(17a^2 + b^2 + 2ab) \\ u &= u(a, b) = \tfrac{1}{4}(17a^2 + b^2 + 34ab) \end{aligned}\right\} \quad (15)$$

Putting 'a' by 4A and 'b' by 4B in the above equations (13) and (15), it is found as

$R = R(A, B) = 68A^2 + 4B^2 + 8AB$

$u = u(A, B) = 68A^2 + 4B^2 + 136AB$

$v = v(A, B) = 272A^2 - 16B^2$

On putting the values of u and v in (2), the non –zero different integrals values of x, y, z, w and R of (1) are found as

$x = x(A, B) = 340A^2 - 12B^2 + 136AB$

$y = y(A, B) = -204A^2 + 20B^2 + 136AB$

$z = z(A, B) = 2(18496A^4 - 64B^4 + 36992A^3B - 2176AB^3) + 1$

$w = w(A, B) = 2(18496A^4 - 64B^4 + 36992A^3B - 2176AB^3) - 1$

$R = R(A, B) = 68A^2 + 4B^2 + 8AB$

Observations:

1. $R(2b,2b) - 320\, t_{4,b} = 0$
2. $x(b+1,b+2) - y(b+1,b+2) - 512t_{4,b} - G_{480b} + PT_6 = 0$
3. $y[1,A(2A^2-1)] + 3[1,A(2A^2-1)] - 32(SO_A)^2 - 160\, So_A = 0$
4. $x(1,A) - y(1,A) + 32\, T_{4,A} - S_{10} = 0 \pmod 3$
5. $3z(4,1) - 3W(A,1)$ is a Nasty number
6. $12y[1, B(2B^2+1)] + 20x[1, B(2B^2+1)] - 4352SO_B \equiv 0 \pmod 2$.
7. $x(2,2) - y(2,2). \equiv 0 \pmod 2$.
8. $x(A,0) - 340t_{4,A} = 0$
9. $x[1, A(A+1)] - y[1, A(A+1)] + 32(P_A)^2 \equiv 0 \pmod 2$.
10. $z(2,2) + w(2,2) \equiv 0 \pmod 2$.

2.5 Pattern: V

Let us take (3) as $u^2 - R^2 = 16R^2 - v^2$

$(u+R)(u-R) = (4R+v)(4R-v)$. (16)

$\frac{u+R}{4R+V} = \frac{4R-V}{U-R} = \frac{A}{B}, B \neq 0$ (17)

This gives us the equations as

$-uA + R(4B+A) - VB = 0$

$uB + R(B-4A) - VA = 0$

By cross multiplication, it leads to

$u = u(A, B) = -A^2 - B^2 - 8AB$

$R = R(A, B) = -A^2 - B^2$

$v = v(A, B) = 4A^2 - 4B^2 - 2AB$

Puting the values of u and v in (2), the non – zero different integral values of x, y, z, w and R of (1) are found as,

$x = x(A, B) = 3A^2 - 6B^2 - 10AB$

$y = y(A, B) = -5A^2 + 5B^2 - 6AB$

$z = z(A, B) = 2[-4A^4 - 4B^4 + 8A^2B^2 - 30A^3B + 30AB^3] + 1$

$w = w(A,B) = 2[-4A^4 - 4B^4 + 8A^2B^2 - 30A^3B + 30AB^3] - 1$

$R = R(A, B) = -A^2 - B^2$

Observations:

1. $R[1,A(A+1)] + (P_A)^2 =$ woodall number
2. $R[A+1,A+2] + P_A \equiv 0 \pmod 3$
3. $5x[1,A(2A^2-1)] + 3[1,A(2A^2-1)] + 68\, SO_A = 0$
4. $x[1,A(A+1)] - 3R[1,A(A+1)] + 10P_A = 6$, a Nasty Number
5. $y(2A,1) + 5R(2A,1) + 28\, t_{4,A} + 12\, P_A = 0$
6. $6x[A(2A^2+1),1] + 10y[A(2A^2+1),1] - 360OH_A = 0$.
7. $x(2,2) - y(2,2).is.$ is a perfect square.
8. $z[2,2] - w(2,2).$ is a square number
9. $R[A,(2A-1)] + T_{12,A} = carol\ number$
10. $x[A(2A^2-1), A(2A^2-1)] + 10SO_A = 0$

3. CONCLUSION

It is interest to see that in (2), the transformations for z and w maybe taken as z = 2u + v and w =2u – v. For this choice, the values of x, y and R are the same as above where as the valueof z and w changes for every method. One may try to see biquadratic Diophantine equations with multivariables (≥ 5) and search for their non-zero distinct integer solutions together with their similar observations.

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