ON CUBIC DIOPHANTINE EQUATION $x^2 + y^2 - xy = 39z^3$

P.Jayakumar[1], J.Meena[2]

[1]*Professor of Mathematics, PeriyarManiammai University, Vallam, Thanajvur -613 403, TN, India.*
[2]*Ph.D. Scholar, Assistant Professor of Mathematics, A.V.V.M. Sri Pushpam College (Autonomous), Poondi -613 503, Thanajvur, TN, India.*

Abstract

Four different methods of the non-zero non-negative solutions of non- homogeneous cubic Diophantine equation $x^2 + y^2 - xy = 39z^2$ are obtained. Some interesting relations among the special numbers and the solutions are exposed.

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Symbols used:

$t_{m,n} = \frac{1}{2}n[(m-2)-(m-4)]$

$p_n^m = \frac{1}{6}[3n^2 + n^3(m-2) - n\,(m-5)]$

$G_n = 2n-1$

1. INTRODUCTION

The number theory is the king of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For a broad review of variety of problems, one may try to see [3-12]. In this work, we are observed another interesting four different methods of the non-zero non-negative solutions the non - homogeneous cubic Diophantine equation $x^2 + y^2 - xy = 39z^3$ Further, some elegant properties among the special numbers and the solutions are observed.

2. DESCRIPTION OF METHOD

Consider the cubic Diophantine equation

$$x^2 + y^2 - xy = 39z^3 \quad (1)$$

Take the linear transformations $x = u + v$, $y = u - v$, $u \neq v \neq 0$) (2)

Using (1) in (2), it gives us $u^2 + 3v^2 = 39z^3$ (3)

If we take $z = z(a, b) = a^2 + 3b^2 = (a + i\sqrt{3}\,b)(a - i\sqrt{3}\,b)$, (4)

where a and b non-zero non- negative different integers, then we solve(1) through dissimilar method of solutions of (1) which are furnished below.

2.1 Method: I

We can write 39 as

$$39 = (6 + i\sqrt{3})(6 - i\sqrt{3}) \quad (5)$$

Using (4) and (5) in (3) and this gives

$(u + i\sqrt{3}\,v)(u - i\sqrt{3}\,v) = (6 + i\sqrt{3})(6 - i\sqrt{3})(a + i\sqrt{3}\,b)^3(a - i\sqrt{3}\,b)^3$

It gives us

$$(u + i\sqrt{3}\,v) = (6 + i\sqrt{3})(a + i\sqrt{3}\,b)^3 \quad (6)$$

$$(u - i\sqrt{3}\,v) = (6 - i\sqrt{3})(a - i\sqrt{3}\,b)^3 \quad (7)$$

Comparing both sides of (6) or (7), we obtain

$u = u(a, b) = 6a^3 - 36ab^2 - 6a^2b + 9b^3$
$v = v(a, b) = a^3 - 6ab^2 + 12a^2b - 18b^3$

In sight of (2), the solutions *x*, *y* are found to be

$$x = x(a, b) = 7a^3 - 42ab^2 + 6a^2b - 9b^3 \quad (8)$$

$$y = y(a, b) = 5a^3 - 30ab^2 - 18a^2b + 27b^3 \quad (9)$$

Hence (4), (8) and (9) gives us two parametric the non-zero different integral values of (1).

Observations:

1. $y(b, b) - x(b, b) - 136P_b^5 + 68t_{4,b} = 0$
2. $z(a, a) - 4t_{4,a} = 0$
3. $x(a, 1) - {}_{14}p_a^5 + t_{4,a} + G_{21a} + 8 = 0$
4. $x(1,a) + 18P_a^5 + 33t_{4,a} - G_{31a} \equiv 0 \pmod 2$
5. $y(1, a) - 54P_a^5 + 57t_{4,a} + G_{9a} \equiv 0 \pmod 3$.
6. $x(3b, 3b) + 10P_b^5 + 5t_{4,b} = 0$.
7. $y(2a, 2a) + 1376P_b^5 - 688t_{4,a} = 0$
8. $z(a, 2a) - 13t_{4,a} = 0$.

2.2method: II

We also write 39 as

$$39 = \frac{(9 + i5\sqrt{3})(9 - i5\sqrt{3})}{4} \quad (10)$$

Using (4) and (10) in (3) and we obtain
$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = [(9+i5\sqrt{3})(9 - i5\sqrt{3})(a + i\sqrt{3}b)^3(a - i\sqrt{3}b)^3]$

Comparing both sides of above, we are found to be

$$(u + i\sqrt{3}v) = \frac{1}{2}[(9 + i5\sqrt{3})(a + i\sqrt{3}b)^3] \quad (11)$$

$$(u - i\sqrt{3}v) = \frac{1}{2}[(9 - i5\sqrt{3})(a - i\sqrt{3}b)^3] \quad (12)$$

Comparing both sides of (11) or (12), we obtain

$$u = u(a, b) = \frac{1}{2}[9a^3 - 54ab^2 - 30a^2b + 45b^3]$$

$$v = v(a, b) = \frac{1}{2}[5a^3 - 30ab^2 + 18a^2b - 27b^3]$$

In true of (2), the values of x, y are given by

$$x = x(a, b) = 7a^3 - 42ab^2 - 6a^2b + 9b^3 \quad (13)$$

$$y = y(a, b) = 2a^3 - 12ab^2 - 24a^2b + 36b^3 \quad (14)$$

Hence (4), (13) and (14) gives us two parametric the non-zero different integral values of (1).

Observations:

1.$y(b, b) - x(b, b) + 88p_b^5 - 44t_{4,b} = 0$

2. $2x(b, b) - 9y(b, b) + 164P_b^5 - 82t_{4,b} = 0$

3. $x(a, 1) - 14P_a^5 + 13t_{4,a} + G_{21a} \equiv 0 \pmod 5$

4. $y(1, b) - 72P_b^5 + t_{52,b} + 23t_{4,b} \equiv 0 \pmod 2$

5. $z(a, 2a) - t_{28,a} - G_{6n} - 1 = 0$

6. $x(a, 3a) + 580P_b^5 - 290t_{4,a} = 0$

7. $z(4a, 4a) - 64t_{4a}, = 0$

8. $y(a, -a) - 4P_b^5 + 2t_{4,a} = 0.$

2.3 Method: III

Let us take (3) as $u^2 + 3v^2 = 39z^3 * 1$ (15)

Consider 1 as $1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$ (16)

Using (4), (5) and (16) in (15) and it gives us

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v) = \frac{1}{4}[(1+i\sqrt{3})(1-i\sqrt{3})(6+i\sqrt{3})(6-i\sqrt{3})(a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3]$$

It gives us

$$(u + i\sqrt{3}v) = \frac{1}{2}[(1+i\sqrt{3})(6 + i\sqrt{3})(a + i\sqrt{3}b) \quad (17)$$

$$(u - i\sqrt{3}v) = \frac{1}{2}[(1 - i\sqrt{3})(6 - i\sqrt{3})(a - i\sqrt{3}b)^3] \quad (18)$$

Comparing both sides of (17) or (18), we obtain

$$u = u(a, b) = \frac{1}{2}[3a^3 - 18ab^2 - 42a^2b + 63b^3]$$

$$v = v(a, b) = \frac{1}{2}[7a^3 - 42ab^2 + 6a^2b - 9b^3]$$

In sight of (2), the values of x, y are given by

$$x = x(a, b) = 5a^3 - 30ab^2 - 18a^2b + 27b^3 \quad (19)$$

$$y = y(a, b) = -2ab^3 + 12ab^2 - 24a^2b + 36b^3 \quad (20)$$

Observations:

1. $y(a, a) - x(a, a) - 76a^2 + 38t_{4,a} = 0$

2. $x(1, b) + 57t_{4,b} - 54P_b^5 - G_{9b} \equiv 0 \pmod 2$

3. $2x(b, b) + 5y(b, b) - 156P_a^5 + 78t_{4,b} = 0$

4. $z(b, b) - 4t_{4,b} = 0$

5. $y(1, b) - 72P_b^5 + G_{12b} + 48t_{4,b} + 1 = 0.$

6. $x(b, 1) = 10P_b^5 + 23t_{4,b} + G_{15,b} \equiv 0 \pmod 2$

7. $z(6a, 6a) - 14t\ t_{4,a} = 0.$

8. $x(2b, 3b) - 3P_b^5 + 3t_{4,b} = 0.$

2.4 Method: IV

Replace (16) by $1 = \frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49}$ (21)

Using (4), (10) and (21) in (15) and this gives us

$$(u + i\sqrt{3}v)(u - i\sqrt{3}v)\frac{1}{4\times 49}[(1+i4\sqrt{3})(1-i\sqrt{3})(9-i5\sqrt{3})(9+i5\sqrt{3})(a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3]$$

This gives us

$$(u + i\sqrt{3}v) = \frac{1}{14}[(1 + i4\sqrt{3})(9 + i5\sqrt{3})(a + i\sqrt{3}b)^3] \quad (22)$$

$$(u - i\sqrt{3}v) = \frac{1}{14}[(1 - i4\sqrt{3})(9 - i5\sqrt{3})(a - i\sqrt{3}b)^3] \quad (23)$$

Comparing both sides of (22) or (23), it gives us

$$u = u(a, b) = \frac{1}{14}[-51a^3 + 306ab^2 - 246a^2b + 369b^3]$$

$$v = v(a, b) = \frac{1}{14}[41a^3 - 246ab^2 - 102a^2b + 153b^3]$$

In true of (2), the values of x, y are given by

$$x = x(a, b) = \frac{1}{7}[-5a^3 + 30ab^2 - 174a^2b + 261b^3 \quad (24)$$

$$y = y(a, b) = \frac{1}{7}[-46a^3 + 276ab^2 - 72a^2b\ 108b^3] \quad (25)$$

Since our intension is to find integer solutions, taking a as 7a and b as 7b in (4), (24) and (25), the related parametricinteger values of(1) are found as

$x = x(a, b) = -245a^3 + 1470ab^2 - 8526a^2b + 12789b^3$

$y = y(a, b) = 2254a^3 + 13524ab^2 - 3528a^2b + 5292b^3$

$z = z(a, b) = 49a^2 + 147b^2$

Observations:

1. $x(a, a) - y(a, a) + 24108P_a^5 - 12054\, t_{4,a} = 0$
2. $z(a, a) - t_{394,a} - G_{97a} - P_a - 1 + t_{4,a} = 0$
3. $x(1, b) + 244 + G_{4263b} - 5578P_b^5 + 1319t_{4,b} = 0$
4. $y(a, 1) - 2508P_a^5 + 5782t_{4,a} - G_{6762a} - 5293 = 0$
5. $z(a, 1) - 49t_{4,a} \equiv 0 \pmod 7$
6. $x(b, 1) + 490\, P_b^5 + 8381\, t_{4,b} - G_{735,b} \equiv 0 \pmod 2$
7. $y(b, 1) - 4508\, P_b^5 + 5782\, t_{4,b} - G_{6762,b} - 5293 = 0$.
8. $z(3a, 3a) - 864\, t_{4,a} = 0$

3. CONCLUSION

Here we observed various process of determining infinitely a lot of non-zero different integer values to the cubic Diophantine equation $x^2+y^2-xy = 39z^3$.One may try to find non-negative integer solutions of the above equations together with their similar observations.

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