EXPONENTIAL-LINDLEY ADDITIVE FAILURE RATE MODEL

K.Rosaiah[1], P.Lakshmi Sravani[2], K.Kalyani[3], D.C.U.Siva Kumar[4]

[1] *Department of Statistics, Acharya Nagarjuna University, Guntur – 522 510, India.*
[2] *Department of Mathematics & Statistics, K.B.N. College (Autonomous), Vijayawada – 520001, India.*
[3] *Research Fellows (UGC), Department of Statistics, Acharya Nagarjuna University, Guntur – 522510, India.*
rosaiah1959@gmail.com

***Abstract*:**
A combination of exponential and Lindley failure rate model is considered and named it as exponential-Lindley additive failure rate model. In this paper, we studied the distributional properties, central and non-central moments, estimation of parameters, testing of hypothesis and the power of likelihood ratio criterion about the proposed model.

Key words: *Exponential distribution, Lindley distribution, ML estimation, Likelihood ratio type criterion.*

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1. INTRODUCTION

Normal distribution and exponential distribution are the basic models exemplified in a number of theoretical results in the theory of distributions. Particularly, exponential distribution is an invariable example for a number of theoretical concepts in reliability studies. It is characterized as constant failure rate (CFR) model also. In case of necessity for an increasing failure rate (IFR) model ordinarily the choice falls on Weibull model with shape parameter more than 1 (>1), in particular taken as 2. Similar in shape, with common characteristics of Weibull, we have Lindley distribution as another IFR model. Lindley distribution has its own prominence as a life testing model. In this paper, we propose to combine an exponential (CFR) model and a Lindley (IFR) model through their hazard functions to get two component series system reliability, which is given as follows:

$$R(x) = e^{-\int_0^x [h_1(x)+h_2(x)]dx} \qquad (1)$$

where $h_1(x)$ and $h_2(x)$ are respectively, the hazard functions of exponential and Lindley distributions.

One such situation is the popular linear failure rate distribution [LFRD]. In that model $h_1(x)$ is taken as a constant failure rate model and $h_2(x)$ is taken as an increasing failure rate (IFR) model with specific choices of exponential for $h_1(x)$ and Weibull with shape 2 for $h_2(x)$. The failure density, the cumulative distribution function, the reliability function and the failure rate of LFRD model are respectively given by

$$f(x;\alpha,\beta) = (\alpha+\beta x)\ \exp\left\{-\alpha x - \frac{\beta}{2}x^2\right\}$$

$$F(x;\alpha,\beta) = 1-\exp\left\{-\alpha x - \frac{\beta}{2}x^2\right\} \quad x>0,\ \alpha,\beta>0$$

$$R(x) = \bar{F}(x;\alpha,\beta) = 1-\mathrm{F}(x;\alpha,\beta) = \exp\left\{-\alpha x - \frac{\beta}{2}x^2\right\}$$

$$h(x;\alpha,\beta) = \alpha + \beta x$$

A number of researchers made an extensive study on LFRD model. Some works in this regard are Bain (1974), Balakrishnan and Malik (1986), Ananda Sen and Bhattacharya (1995), Mohie El-Din *et al*. (1997), Ghitany and Kotz (2007), ABD EL –Baset A.Ahmad (2008), Mahmoud and Al-Nagar (2009).

Kantam and Priya (2011) considered an additive life testing model combining a CFR and DFR model with DFR generated from a Weibull model of shape parameter < 1. Srinivasarao *et al.* (2013a) studied the properties, estimation and testing of linear failure rate model with exponential and half – logistic distribution. Srinivasarao *et al.* (2013b) have discussed the distributional properties, estimation of parameters and testing of hypothesis for additive failure rate model combining exponential and gamma distributions. Rosaiah *et al.* (2014) studied the estimation of parameters, testing of linear failure rate with exponential and modified Weibull distribution. Srinivas (2015) considered an additive failure rate model combining exponential and generalized half logistic distributions.

The probability density function (PDF), cumulative distribution function (CDF) and hazard function (HF) of the exponential distribution with scale parameter λ are respectively given by

$$f_1(x;\lambda) = \lambda e^{-\lambda x} \quad ; x\geq 0, \lambda>0$$

$$F_1(x;\lambda) = 1-e^{-\lambda x} \quad ; x\geq 0, \lambda>0$$

$$h_1(x;\lambda) = \lambda$$

The Lindley distribution was originally proposed by Lindley (1958) in the context of Bayesian statistics, a counter example of fudicial statistics. The PDF, CDF and HF of the Lindley distribution are respectively given by

$$f_2(x;\theta) = \frac{\theta^2(1+x)e^{-x\theta}}{(1+\theta)}, \quad x \geq 0, \theta > 0$$

$$F_2(x;\theta) = 1 - e^{-\theta x}\left[1 + \frac{\theta x}{(\theta+1)}\right], \quad x \geq 0, \theta > 0$$

$$h_2(x;\theta) = \frac{\theta^2(1+x)}{1+(\theta(1+x))}, \quad x \geq 0, \theta > 0$$

where θ is the scale parameter.

The Lindley distribution is a two-component mixture of exponential distribution (with scale parameter θ) and gamma distribution (with shape parameter 2 and scale parameter θ) with mixing proportion θ/(θ+1). In the context of reliability studies, Ghitany *et al.* (2007) studied the structural and inferential properties of the Lindley distribution as well as its application. Ghitany *et al.* (2007) showed that the PDF of the Lindley distribution is uni-model when $0<\theta<1$, and is decreasing when $\theta \geq 1$. They have also shown that the shape of the hazard function is an increasing function and hence it is an IFR model. Al-Mutairi *et al.* (2013) have studied the inferences on stress-strength reliability from Lindley distribution.

By substituting $h_1(x;\lambda)$ and $h_2(x;\theta)$ in equation (1), we get the reliability of the two component series system

$$R(x) = \left[\frac{(\theta x+\theta+1)e^{-x(\lambda+\theta)}}{(\theta+1)}\right], \quad x \geq 0, \lambda > 0, \theta > 0 \qquad (2)$$

We obtain the probability density function corresponding to the reliability function given in (2) and named it as exponential-Lindley additive failure rate model (ELAFRM). The distributional properties, graphical natures of ELAFRM for different choices of λ, θ and the estimation of parameters are discussed in Section 2. Testing of hypothesis and the power of likelihood ratio criterion about the proposed model are presented in Section 3. Conclusions are given in Section 4.

2. DISTRIBUTIONAL PROPERTIES AND ESTIMATION OF PARAMETERS

2.1. Distributional Properties

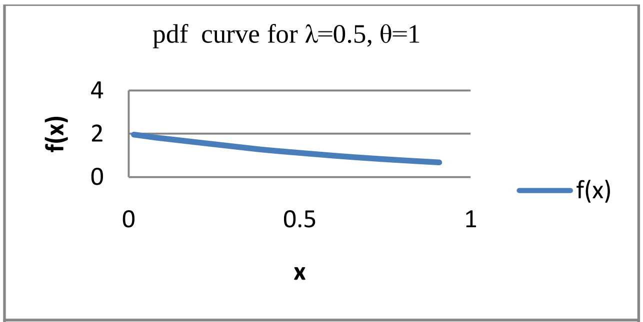
The probability density function, cumulative distribution function and hazard function of exponential–Lindley additive failure rate model (ELAFRM) from Equation (2) are respectively given by,

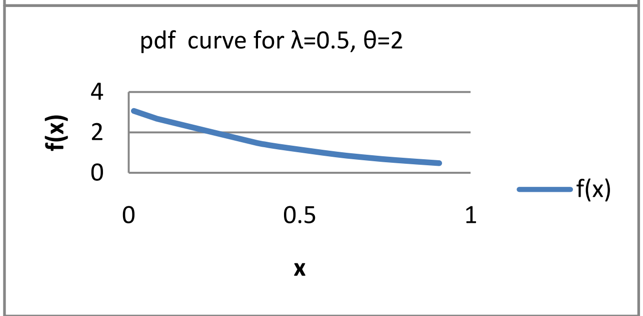
$$g(x;\lambda,\theta) = \frac{e^{-x(\lambda+\theta)}\left[\left((\theta x+\theta+1)(\theta+\lambda)\right)-\theta\right]}{(\theta+1)}, \quad x \geq 0, \ \lambda > 0, \ \theta > 0 \qquad (3)$$

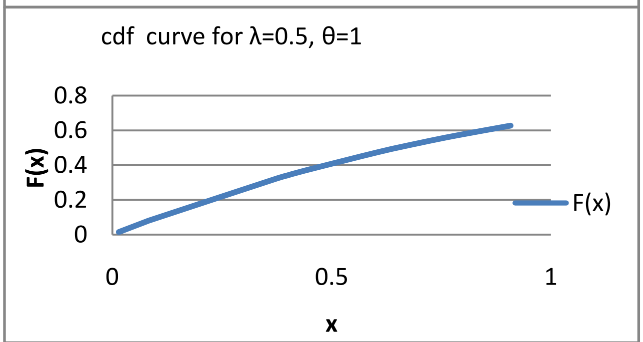
$$G(x;\lambda,\theta) = 1 - \left[\frac{(\theta x+\theta+1)e^{-x(\lambda+\theta)}}{(\theta+1)}\right], \quad x \geq 0, \ \lambda > 0, \ \theta > 0 \qquad (4)$$

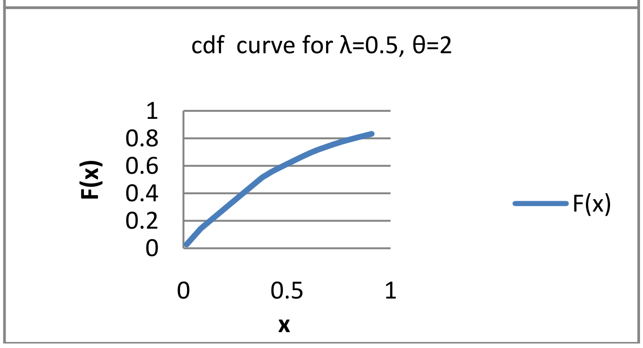
$$h(x;\lambda,\theta) = \frac{\left[(\theta x+\theta+1)(\theta+\lambda)-\theta\right]}{(\theta x+\theta+1)}, \quad x \geq 0, \ \lambda > 0, \ \theta > 0 \qquad (5)$$

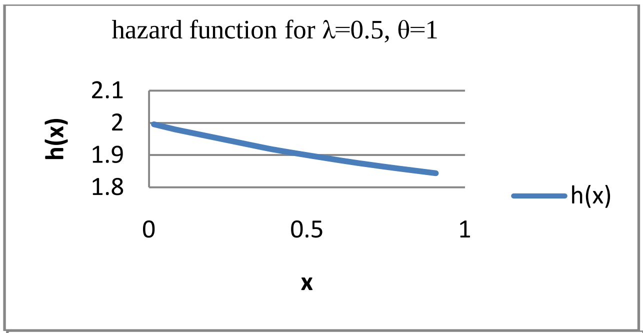
The shapes of the probability density function, distribution function, hazard function are as shown in the following graphs for λ= 0.5 and θ=1, 2. The hazard function appears to be a decreasing function, hence ELAFRM is a decreasing failure rate (DFR) model.

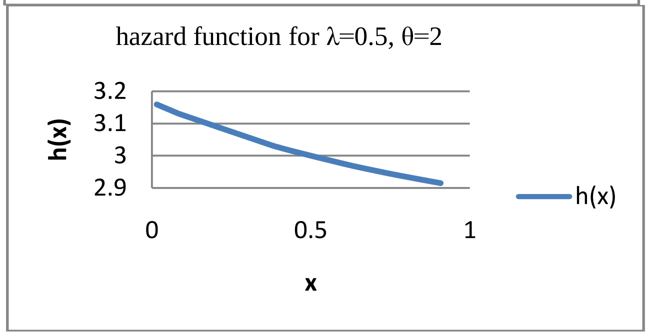












Central and non-central moments of ELAFRM are as follows:

Non–Central moments:

$$\mu_1^1 = \frac{\lambda(\theta+1)+\theta(\theta+2)}{(\theta+1)(\lambda+\theta)^2}, \qquad \mu_2^1 = \frac{2\lambda(\theta+1)+2\theta(\theta+3)}{(\theta+1)(\lambda+\theta)^3}$$

$$\mu_3^1 = \frac{6\lambda(\theta+1)+6\theta(\theta+4)}{(\theta+1)(\lambda+\theta)^4}, \qquad \mu_4^1 = \frac{24\lambda(\theta+1)+24\theta(\theta+5)}{(\theta+1)(\lambda+\theta)^5}$$

CENTRAL MOMENTS:

$$\mu_1 = \frac{[\lambda(\theta+1)+\theta(\theta+2)]}{(\theta+1)(\lambda+\theta)^2}$$

$$\mu_2 = \frac{1}{(\theta+1)(\lambda+\theta)^3}\left[2\lambda(\theta+1)+2\theta(\theta+3)-\left[\frac{[\lambda(\theta+1)+\theta(\theta+2)]^2}{(\theta+1)(\lambda+\theta)}\right]\right]$$

$$\mu_3 = \frac{1}{(\theta+1)(\lambda+\theta)^4}\left\{\begin{array}{l}(6\lambda(\theta+1)+6\theta(\theta+4)) \\ -3\left[\frac{(2\lambda(\theta+1)+2\theta(\theta+3))(\lambda(\theta+1)+\theta(\theta+2))}{(\theta+1)(\lambda+\theta)}\right]+2\left[\frac{(\lambda(\theta+1)+\theta(\theta+2))^3}{(\theta+1)^2(\lambda+\theta)^2}\right]\end{array}\right\}$$

$$\mu_4 = \left[\frac{24\lambda(\theta+1)+24\theta(\theta+5)}{(\theta+1)(\lambda+\theta)^5}\right]-4\left[\frac{6\lambda(\theta+1)+6\theta(\theta+4)}{(\theta+1)(\lambda+\theta)^4}\right]\left[\frac{[\lambda(\theta+1)+\theta(\theta+2)]}{(\theta+1)(\lambda+\theta)^2}\right]+$$
$$6\left[\frac{2\lambda(\theta+1)+2\theta(\theta+3)}{(\theta+1)(\lambda+\theta)^3}\right]\left[\frac{[\lambda(\theta+1)+\theta(\theta+2)]}{(\theta+1)(\lambda+\theta)^2}\right]^2-3\left[\frac{[\lambda(\theta+1)+\theta(\theta+2)]}{(\theta+1)(\lambda+\theta)^2}\right]^4$$

2.2. Maximum Likelihood Estimation (MLE)

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n drawn from ELAFRM with density function g(x;λ, θ), then the likelihood function of $(x_1, x_2, \ldots, x_n)$ is given by

$$L = \prod_{i=1}^{n} g(x_i;\lambda,\theta) \qquad \Rightarrow L \propto \prod_{i=1}^{n}\left[\frac{e^{-x_i(\lambda+\theta)}((\theta x_i+\theta+1)(\theta+\lambda)-\theta)}{(\theta+1)}\right]$$

$$\Rightarrow \log L = -\sum_{i=1}^{n} x_i(\lambda+\theta) + \sum_{i=1}^{n}\log\Big(\big((\theta x_i+\theta+1)(\theta+\lambda)\big)-\theta\Big) - n\log(\theta+1)$$

The MLE's $\hat{\lambda}, \hat{\theta}$ of λ and θ can be obtained by simultaneously solving the following ML equations,

$$\frac{\partial \log L}{\partial \lambda} = 0$$

$$\Rightarrow -\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{(\theta x_i+\theta+1)}{\left[\big((\theta x_i+\theta+1)(\theta+\lambda)\big)-\theta\right]} = 0$$

$$\frac{\partial \log L}{\partial \theta} = 0$$

$$\Rightarrow -\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\left[\frac{(\theta x_i+\theta+1)+\big((\theta+\lambda)(x_i+1)\big)-1}{\left[\big((\theta x_i+\theta+1)(\theta+\lambda)\big)-\theta\right]}\right] - \frac{n}{(\theta+1)} = 0$$

The asymptotic variances and covariance of the estimators of the parameters are obtained using the following elements of the information matrix

$$I_{11} = -E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right) = -E\left[-\sum_{i=1}^{n}\left[\frac{(\theta x_i+\theta+1)^2}{\left[\big((\theta x_i+\theta+1)(\lambda+\theta)\big)-\theta\right]^2}\right]\right]$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 \log L}{\partial \theta \partial \lambda}\right) = -E\left[-\sum_{i=1}^{n}\left[\frac{\theta(x_i+1)(\theta x_i+\theta+2)}{\left[\big((\theta x_i+\theta+1)(\lambda+\theta)\big)-\theta\right]^2}\right]\right]$$

$$I_{22} = -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -E\left[-\sum_{i=1}^{n}\left[\frac{\left[(\theta x_i+\theta+1)+\big((\theta+\lambda)(x_i+1)\big)-1\right]^2 - \left[(\theta x_i+\theta+1)\big((\theta+\lambda)-\theta\big)(2x_i-2)\right]}{\left[\big((\theta x_i+\theta+1)(\theta+\lambda)\big)-\theta\right]^2}\right] + \frac{n}{\left((\theta+1)^2\right)}\right]$$

The estimated asymptotic variance-covariance matrix of the MLEs is given by

$$D(\hat{\lambda}, \hat{\theta}) = [\hat{I}_{11}\hat{I}_{22} - \hat{I}_{12}^2]^{-1}\begin{bmatrix}\hat{I}_{22} & -\hat{I}_{12}\\ -\hat{I}_{21} & \hat{I}_{11}\end{bmatrix}$$

$$where\ \hat{I}_{11} = -E\left(\frac{\partial^2 \log L}{\partial \lambda^2}\right)_{\lambda=\hat{\lambda},\theta=\hat{\theta}},\ \hat{I}_{22} = -E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right)_{\lambda=\hat{\lambda},\theta=\hat{\theta}},\ \hat{I}_{12} = \hat{I}_{21} = -E\left(\frac{\partial^2 \log L}{\partial \theta \partial \lambda}\right)_{\lambda=\hat{\lambda},\theta=\hat{\theta}}$$

3. LIKELIHOOD RATIO TYPE CRITERION AND CRITICAL VALUES

Let us designate our proposed ELAFRM as null population say P_0 and exponential distribution as alternate population say P_1. We propose a null hypothesis H_0 : "A given sample belongs to the population P_0" against an alternative hypothesis H_1 : "The sample belongs to population P_1".

Let L_1, L_0 respectively stand for the likelihood functions of the sample with population P_1 and P_0. Both L_1 and L_0 contains the respective parameters of the populations. The given sample is used to get the parameters of P_1, P_0, so that for the given sample, the value of L_1/L_0 is now estimated. If H_0 is true, L_1/L_0 must be small, therefore for accepting H_0 with a given degree of confidence, L_1/L_0 is compared with a critical value with the help of the percentiles in the sampling distribution of L_1/L_0.

We have seen in Section 2.2, how to get the estimates of parameters. But the sampling distribution of L_1/L_0 is not analytical, we therefore resorted to the empirical sampling distribution through simulation. We have generated 3000 random samples of size n=5(1)10 from the population P_0 with various parameter combinations and obtain the values of L_1, L_0 along with the estimates of respective parameters for each sample.

The percentiles of L_1/L_0 at various probabilities are computed and are given in Tables 1 and 2.

Table 1 Percentiles of L_1/L_0 for $\lambda = 0.5, \theta = 1$

n	**0.99**	**0.975**	**0.95**	**0.05**	**0.025**	**0.01**
5	2.4010	1.6795	1.3022	0.0332	0.0319	0.0296
6	2.7012	1.6651	1.1134	0.0169	0.0154	0.0146
7	2.1296	1.4004	0.9297	0.0090	0.0081	0.0074
8	1.8671	1.1157	0.6764	0.0047	0.0042	0.0037
9	1.6571	0.9964	0.5684	0.0024	0.0021	0.0019
10	1.1160	0.7191	0.4074	0.0013	0.0012	0.0010

Table Percentiles of L_1/L_0 for $\lambda = 0.5, \theta = 2$

n	**0.99**	**0.975**	**0.95**	**0.05**	**0.025**	**0.01**
5	0.3032	0.2052	0.1768	0.0044	0.0042	0.0040
6	0.2196	0.1384	0.0916	0.0015	0.0014	0.0013
7	0.1168	0.0760	0.0546	0.0005	0.0005	0.0005
8	0.0636	0.0422	0.0257	0.0002	0.0002	0.0002
9	0.0401	0.0252	0.0145	0.0001	0.0001	0.0001
10	0.0182	0.0124	0.0069	0.0000	0.0000	0.0000

4. POWER OF LIKELIHOOD RATIO TYPE CRITERION

In testing of hypothesis, we know that the power of a test statistic is the complementary probability of accepting a false hypothesis at a given level of significance. Let us conventionally fix 5% level of significance. The percentiles given in Tables 1 and 2 under the column 0.05 are the critical values. We generate 3000 random samples of sizes 5(1)10 from the population P_1 namely exponential. At this sample we find the ML estimates of the parameters of P_1 and P_0 using the respective probability models. Accordingly, we get the estimates of L_1, L_0 for the sample from P_1.

Over repeated simulation runs, we get the proportion of values of L_1/L_0 that fall below the respective critical values of Tables 1 and 2. These proportions would give the value of β, the probability of type II error. If the test statistic has a discriminating power, β must be small so that the power $l- \beta$ must be large. Various power values are given in Tables 3 and 4. We see that as n increases and hence $1- \beta$ increases. We conclude that as long as n increases the power of the likelihood criterion increases. We therefore conclude that exponential model can be a reasonable alternative to the proposed ELAFR model in small samples.

Table 3 Power of Likelihood Ratio criterion for $\lambda = 0.5, \theta = 1$

n	***0.975***	***0.950***
5	0.974	0.949
6	0.975	0.955
7	0.976	0.953
8	0.978	0.955
9	0.978	0.957
10	0.978	0.958

Table 4 Power of Likelihood Ratio criterion for $\lambda = 0.5, \theta = 2$

n	***0.975***	***0.950***
5	0.969	0.968
6	0.978	0.977
7	0.983	0.980
8	0.984	0.983
9	0.991	0.987
10	0.991	0.988

5. CONCLUSIONS

Exponential and Lindley failure rate models are considered for reliability studies and are named as exponential-Lindley additive failure rate model. The distributional properties, estimation of parameters, testing of hypothesis and the power of likelihood ratio criterion about the proposed model are discussed and the results are presented.

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