# NUMERICAL SIMULATION ON LAMINAR CONVECTION FLOW AND HEAT TRANSFER OVER AN ISOTHERMAL VERTICAL PLATE EMBEDDED IN A SATURATED POROUS MEDIUM

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## Abstract

*A numerical algorithm is presented for studying laminar convection flow and heat transfer over an isothermal vertical horizontal plate embedded in a saturated porous medium. By means of similarity transformation, the original nonlinear partial differential equations of flow are transformed to a pair of nonlinear ordinary differential equations. Subsequently they are reduced to a first order system and integrated using Newton Raphson and adaptive Runge-Kutta methods. The computer codes are developed for this numerical analysis in Matlab environment. Velocity and temperature profiles are illustrated graphically. Heat transfer parameters are derived.*

***Keywords**: Free Convection, Heat Transfer, Isothermal Vertical Plate, Matlab, Numerical Simulation, Porous Medium.*

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**List of Symbols**$a_1$, $a_2$ initial values eq (21)

f function defined in eq (13)
g gravitational acceleration, 9.81 m/s$^2$
K permeability of the fluid, dimensionless
$Nu_x$ Nusselt number at x, dimensionless
$q_w$ heat flux of the plate, W/m$^2$
Ra Rayleigh number, dimensionless
T temperature, K
$T_w$ surface temperature, K
$T_\infty$ free streams temperature, K
u velocity component in x, m/s
v velocity component in y, m/s
x coordinate from the leading edge, m
y coordinate normal to plate, m
$z_1$, $z_2$, $z_3$, $z_4$ variables, eq (18)

**Greek Symbols**

$\theta$ function defined in eq (12), dimensionless
$\beta$ coefficient of thermal expansion, 1/K
$\delta$ boundary layer thickness, m
$\alpha_a$ apparent thermal diffusivity, m$^2$/s
$\mu_f$ dynamic viscosity, N.s/m$^2$
$\nu_f$ kinematic viscosity, m$^2$/s
$\eta$ similarity variables, eq (14)
$\psi$ stream function, m$^2$/s
$\rho_f$ density, kg/m$^3$

## 1. INTRODUCTION

There have been a number of studies on natural convection over an isothermal vertical plate in a porous medium due to its relevance to a variety of industrial applications and naturally occurring processes, such as heat insulation by fibrous materials, spreading of pollutants and convection in the earth's mantle, storage of agricultural products such as fruits, vegetables and grains, ground water hydrology, disposal of wastes, petroleum reservoir engineering, mineral extraction and oil recovery system, recovery of water for drinking and irrigation, salt water encroachment into fresh water reservoirs, in biophysics-life processes such as flow in the lungs and kidneys etc.

The earliest analytical investigation in this regard was a similarity analysis of the boundary layer equations by P. Cheng and W. J. Minkowycz, J. [1]. The problem is also discussed by several authors [2-5]. The problem was discussed in several text books [6-10].

In the present numerical investigation, a simple accurate numerical simulation of laminar free-convection flow and heat transfer over an isothermal horizontal plate is developed.

The paper is organized as follows: Mathematical model of the problem, its solution procedure, development of code in Matlab, interpretation of the results, comparison with Previous work.

## 2. MATHEMATICAL MODEL

We consider the natural convection about an isothermal vertical impermeable flat plate immersed in a saturated porous medium with constant permeability K. We assume the natural convection flow to be steady, laminar, two-dimensional, having no dissipation, and the flow through the porous medium is governed by Darcy's law, with constant properties, including density, with one exception: the density difference $\rho - \rho_\infty$ is to be considered since it is this density difference between the inside and the outside of the boundary layer that gives rise to buoyancy force and sustains flow, known in the literature as *Boussinesq approximation*. We take the direction along the plate to be *x*, and the direction normal to surface to be *y*, as shown in Fig 1.

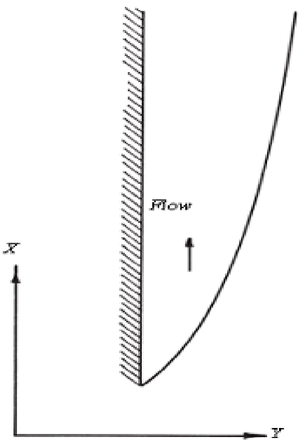


**Fig. 1** Physical Model and its coordinate system

The equations governing the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad 1$$

$$u = \frac{\beta g K \rho_f (T - T_\infty)}{\mu_f} \qquad 2$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_a \frac{\partial^2 T}{\partial y^2} \qquad 3$$

The boundary conditions on the solution are:

At y=0: v=0, $T=T_w$
For large y: $u \to 0$, $T \to T_\infty$ 4

The continuity equation (1) is automatically satisfied through introduction of the stream function:

$$u \equiv \frac{\partial \psi}{\partial y} \qquad v \equiv -\frac{\partial \psi}{\partial x} \qquad 5$$

Eqs (2) and (3) can be written in terms of stream function:

$$\frac{\partial \psi}{\partial y} = \frac{\beta g K \rho_f (T - T_\infty)}{\mu_f} \qquad 6$$

$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \alpha_a \frac{\partial^2 T}{\partial y^2} \qquad 7$$

From (6), we can write

$$\frac{\partial^2 T}{\partial y^2} = \frac{\beta g K \rho_f}{\mu_f}\frac{\partial T}{\partial y} \qquad 8$$

From scale analysis, it can be shown that

$$\frac{\delta}{x} = o(Ra^{-0.5}) \qquad 9$$

And

$$\psi = o(\alpha_a Ra^{0.5}) \qquad 10$$

Where

$$Ra = \frac{\beta g K \rho_f (T_w - T_\infty)}{\alpha_a \mu_f} = \frac{\beta g K (T_w - T_\infty) x}{\alpha_a \nu_f} \qquad 11$$

is Darcy-modified Rayleigh number.

We expect that in the boundary layer, the stream function and temperature profiles are similar:

$$\frac{T - T_\infty}{T_w - T_\infty} = \theta\left(\frac{y}{\delta}\right) = \theta\left(\frac{y}{x} Ra^{0.5}\right) = \theta(\eta) \qquad 12$$

and

$$\frac{\psi}{\alpha_a Ra^{0.5}} = f(\eta) \qquad 13$$

where η is the similarity solution given by

$$\eta = \frac{y}{x} Ra^{0.5} \qquad 14$$

With the help of eqs (12) and (13), eqs (8) and (7) can be written as (with a prime denoting differentiation with respect to η)

$$f'' = \theta' \qquad 15$$

and

$$\theta'' = -\frac{f\theta'}{2} \qquad 16$$

Hence the velocity boundary layer problem is reduced to an ordinary differential equation (15) and energy eq (3) is also reduced to an ordinary differential equation (16). This confirms the assumption that velocity and temperature profiles are similar. The appropriate boundary conditions (4) are now:

y = 0: v = 0, $T=T_w$ i.e., at η = 0: f = 0, θ = 1

large y: $u \to 0$, $T \to T_\infty$ i.e., η large: $f' \to 0$, $\theta \to 0$ 17

## 3. SOLUTION PROCEDURE

Eqs (15) and (16) are simultaneous nonlinear ordinary differential equations for the velocity and temperature functions, $f'$ and θ. No analytical solution is known, so numerical integration is necessary. Values of f and θ at the surface of the plate (y = 0), and that of $f'$ and $\theta$ far away from the surface (y → ∞) are known.

*3.1 Reduction of Equations to First-order System*

This is done easily by defining new variables:

$$z_1 = f$$
$$z_2 = z_1' = f'$$
$$z_3 = \theta$$
$$z_4 = z_3' = \theta' \quad 18$$

Therefore from eqs (15) and (16), we get the following set of differential equations

$$z_1' = f'$$
$$z_2' = z_1'' = f'' = \theta' = z_4$$
$$z_3' = z_4$$
$$z_4' = z_3'' = \theta'' = -\frac{f\theta'}{2} = -\frac{z_1 z_4}{2} \quad 19$$

with the following boundary conditions:

$$z_1(0) = f(0) = 0$$
$$z_2(\infty) = z_1'(\infty) = f'(\infty) = 0$$
$$z_3(0) = \theta(0) = 1$$
$$z_3(\infty) = \theta(\infty) = 0 \quad 20$$

Eqs (15) and (16) are second-order each and are replaced by two first-order equations (21).

*3.2 Solution to Initial Value Problems*

To solve eqs (21), we denote the two unknown initial values by $a_1$ and $a_2$, the set of initial conditions is then:

$$z_1(0) = f(0) = 0$$
$$z_2(0) = z_1'(0) = f'(0) = a_1$$
$$z_3(0) = \theta(0) = 1$$
$$z_4(0) = z_3'(0) = \theta'(0) = a_2 \quad 21$$

If eqs (19) are solved with adaptive Runge-Kutta method using the initial conditions in eq (21), the computed boundary values at $\eta = \infty$ depend on the choice of $a_1$ and $a_2$ respectively. We express this dependence as

$$z_2(\infty) = z_1'(\infty) = f'(\infty) = f_1(a_1)$$
$$z_3(\infty) = \theta(\infty) = f_2(a_2) \quad 22$$

The correct choice of $a_1$ and $a_2$ yields the given boundary conditions at $\eta = \infty$; that is, it satisfies the equations

$$f_1(a_1) = 0$$
$$f_2(a_2) = 0 \quad 23$$

These nonlinear equations can be solved by the Newton-Raphson method. A value of 10 is fine for infinity, even if we integrate further nothing will change.

*3.3 Program Details*

This section describes a set of Matlab routines for the solution of eqs (19) along with the initial conditions (21). They are listed in Table 1.

**Table 1.** A set of Matlab routines used sequentially to solve Equations (24).

| Matlab code | Brief Description |
|---|---|
| deqs.m | Defines the differential equations (19). |
| incond.m | Describes initial values for integration, $a_1$ and $a_2$ are guessed values, eq (21) |
| runKut5.m | Integrates as initial value problem using adaptive Runge-Kutta method. |
| residual.m | Provides boundary residuals and approximate solutions. |
| newtonraphson.m | Provides correct values $a_1$ and $a_2$ using approximate solutions from residual.m |
| runKut5.m | Again integrates eqs (19) using correct values of $a_1$ and $a_2$. |

The final output of the code runKut5.m gives the tabulated values of $f$, $f'$ for velocity profile, and $\theta$ and $\theta'$ for temperature profiles as function of η.

## 4. INTERPRETATION OF THE RESULTS

Physical quantities are related to the dimensionless functions $f$ and $\theta$ through eqs (12), (13) and (14). $f$ and $\theta$ are now known.

### 4.1 Computed Values of the Parameters

Some accurate initial values from this computation are listed in Table 2. These theoretical computations are in good agreement with results published in literatures [10].

**Table 2**. Computed values from eqs (15) & (1)

| η | $f$ | $f'$ | $\theta$ | $\theta'$ |
|---|---|---|---|---|
| 0.00000 | 0.00000 | 1.00000 | 1.00000 | -0.44390 |
| 0.10000 | 0.09778 | 0.95565 | 0.95565 | -0.44281 |
| 0.56453 | 0.49467 | 0.75551 | 0.75551 | -0.41263 |
| 1.06139 | 0.82149 | 0.56527 | 0.56527 | -0.34971 |
| 1.58921 | 1.07474 | 0.40121 | 0.40121 | -0.27178 |
| 2.17789 | 1.26863 | 0.26537 | 0.26537 | -0.19212 |
| 2.81160 | 1.40304 | 0.16583 | 0.16583 | -0.12561 |
| 3.44239 | 1.48601 | 0.10208 | 0.10208 | -0.07956 |
| 4.03844 | 1.53462 | 0.06386 | 0.06386 | -0.05070 |
| 4.65045 | 1.56554 | 0.03914 | 0.03914 | -0.03154 |
| 5.29663 | 1.58523 | 0.02316 | 0.02316 | -0.01895 |
| 5.99060 | 1.59747 | 0.01305 | 0.01305 | -0.01091 |
| 6.74612 | 1.60476 | 0.00687 | 0.00687 | -0.00596 |
| 7.57896 | 1.60881 | 0.00325 | 0.00325 | -0.00305 |
| 8.50860 | 1.61079 | 0.00125 | 0.00125 | -0.00144 |
| 9.56051 | 1.61149 | 0.00023 | 0.00023 | -0.00062 |
| 10.00000 | 1.61154 | 0.00000 | 0.00000 | -0.00043 |

### 4.2 Dimensionless Stream Function and Temperature Profiles

Variations of f and $\theta$ obtained from the present computation are shown in Fig 2.

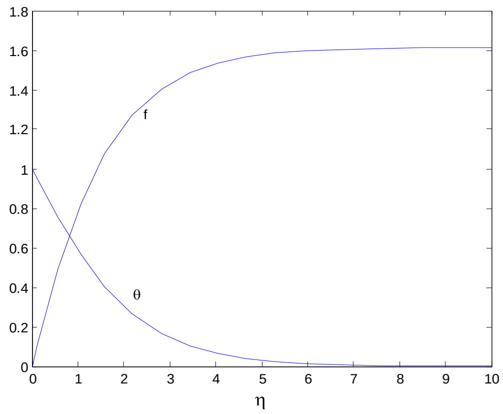


**Fig. 2** Solutions of f and $\theta$

### 4.3 Comparison With Experiments

Fig 3 compares the dimensionless temperature profile θ, with the experiments of Evans and Plumb [11]. The agreement is excellent.

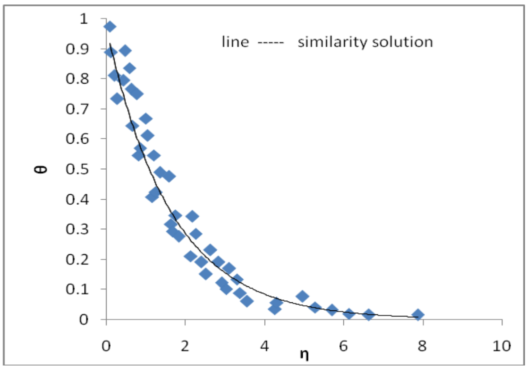


**Fig 3.** Comparison of the computed dimensionless temperature profile with experimental data.

### 4.4 Heat Transfer Parameters

Besides the velocity and temperature distributions, it is often desirable to compute other physically important quantities associated with the convection flow.

The heat transfer rate at the wall is given by

$$q_w = -k \frac{\partial T}{\partial y}\bigg|_{y=0} \tag{24}$$

Hence, using eq (12)

$$\frac{q_w x}{k_a (T_w - T_\infty)} = -Ra^{0.5} \theta'\big|_0 \tag{25}$$

From Table 2, we see that

$$\theta'\big|_0 = -0.4439$$

Hence,

$$Nu_x = 0.444 Ra^{0.5} \tag{26}$$

where, $Nu_x$ is the local Nusselt number based on x. Fig 4 illustrates the comparison of the correlation (26) with the experimental data of Evans and Plumb [11].

The mean heat transfer rate is then given by using

$$\langle q_w \rangle = \frac{1}{L}\int_0^L q_w dx \tag{27}$$

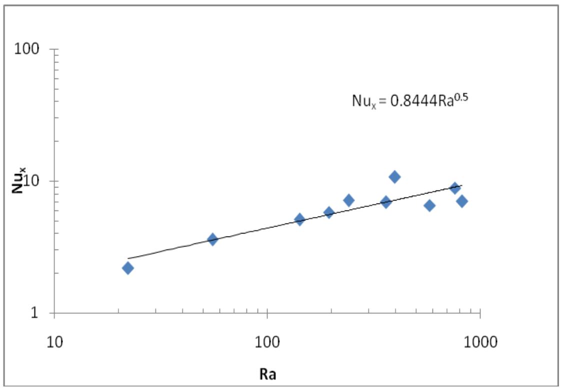


**Fig 4.** Comparison of the computed correlation with experimental data.

Eqs (26) and (27) gives

$$Nu_L = 0.888 Ra_L^{\ 0.5} \tag{28}$$

where $Nu_L$ is the mean Nusselt number based on the length of the plate, L and $Ra_L$ is Darcy-modified Rayleigh number based on L. These empirical correlations are found in the text books [6-10].

## 5. CONCLUSION

In the present numerical simulation, laminar convection flow and heat transfer over an isothermal vertical horizontal plate embedded in a saturated porous medium is presented.

Details of the solution procedure of the nonlinear partial differential equations of flow are discussed. The computer codes are developed for this numerical analysis in Matlab environment. Velocity profile and temperature profiles are computed using these codes. The computed and experimental velocity and temperature distributions are in very good agreement with results published in literatures. Heat transfer parameters are derived. A good agreement between the present results and the past indicates that the developed numerical simulation as an efficient and stable numerical scheme in natural convection.

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