INTEGRAL SOLUTIONS OF THE TERNARY CUBIC EQUATION

$$5(x^2+y^2)-9xy+x+y+1=35z^3$$

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Abstract

The ternary cubic equation $5(x^2+y^2)-9xy+x+y+1=35z^3$ *is considered for determining its non-zero distinct integral solutions Employing the linear transformations x=u+v,y=u-v (u≠v≠0),and employing the meyhod of factorization in complex conjugates, different patterns of integral solutions to the ternary cubic equation under consideration are obtained.. In each pattern, interesting relations among the solutions, some special polygonal , pyramidal numbers and central pyramidal numbers are exhibited.*

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1. INTRODUCTION

Diophantine equations are numerously rich because of its variety. The determination of integral solutions for cubic(homogeneous or non-homogeneous) diophantine equations with three variables has been an interest to mathematicians since antiquity as can be seen from [1-3].In this context one may refer [4-24]. In this Communication, the non-homogeneous ternary cubic diophantine equation represented by $5(x^2+y^2)-9xy+x+y+1=35z^3$ is considered for its non-zero distinct integral solutions. A few interesting relations between special polygonal numbers and pyramidal numbers are exhibited.

2. METHOD OF ANALYSIS:

The ternary cubic diophantine equation under consideration is

$$5(x^2+y^2)-9xy+x+y+1=35z^3 \qquad (1)$$

Introduction of the transformations

$$x=u+v\,,\ y=u-v \qquad (2)$$

in (1) leads to

$$(u+1)^2+19v^2=35z^3 \qquad (3)$$

Equation (3) is solved through different methods and thus, we obtain different patterns of solutions to (1)

2.1 Method 1

Take $z=a^2+19b^2$ (4)

Write $35=(4+i\sqrt{19})(4-i\sqrt{19})$ (5)

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u+1)+i\sqrt{19}v=(4+i\sqrt{19})(a+i\sqrt{19}b)^3 \qquad (6)$$

Equating real and imaginary parts of (6) on both sides we get

$$u+1=4a^3-57a^2b-228ab^2+361b^3$$
$$v=a^3+12a^2b-57ab^2-76b^3$$

Substituting the values of u,v in (2) the non-zero distinct integral solutions to (1) are given by

$$x=5a^3-45a^2b-285ab^2+285b^3-1$$
$$y=3a^3-69a^2b-171ab^2+437b^3-1$$
$$z=a^2+19b^2$$

Properties:

1) $x(a,1)-30P^3_{a-1}+t_{92,a}\equiv -40(\mathrm{mod}-324)$
2) $y(a,1)-x(a,1)+3CP^3_a+t_{50,a}\equiv 60(\mathrm{mod}92)$
3) $4[x(a,1)-y(a,1)+8z(a,1)-19S_a+82t_{4,a}]+76$ is a cubical integer

4) $23z(a,1)-y(a,1)+2CP_a^9-138t_{3,a}\equiv 1(\bmod 124)$

5) $10[y(a+1,a)-360P_{a-1}^3-280CP_a^3+140PR_a+1]$ is a perfect square

6) $x(1,B)+570P_b^5+45PR_b+525t_{4,b}$ is a perfect square

7) $2[z(1,b)-1]-19(CS_b+Gno_b)=0$

8) $z(a,1)-2t_{5,a}+t_{6,a}\equiv 0(\bmod 19)$

Note:
Re-writing (5) as

$$35=(-4+i\sqrt{19})(-4-i\sqrt{19})$$

and proceeding as in method1, the non-zero distinct integral solutions to (1) are given by

$$x=-3a^3-69a^2b+171ab^2+437b^3-1$$
$$y=-5a^3-45a^2b+285ab^2+285b^3-1$$
$$z=a^2+19b^2$$

2.2 Method 2:

Equation (3) can be written as

$$(u+1)^2+19v^2=35z^3.1$$

Write '1' as

$$1=\frac{(9+i\sqrt{19})(9-i\sqrt{19})}{10^2}$$

Define

$$(u+1+i\sqrt{19}v)=\frac{(4+i\sqrt{19})(a+i\sqrt{19}b)^3(9+i\sqrt{19})}{10} \quad (7)$$

Equating real and imaginary parts, we have

$$u+1=\frac{1}{10}[17a^3-741a^2b-969ab^2+4693b^3] \quad (8)$$

$$v=\frac{1}{10}[13a^3+51a^2b-741ab^2-323b^3] \quad (9)$$

Since our aim is to find integer solutions, assuming a=10A and b=10B in (4),(8) and (9) and substituting the values of u,v in (2), we obtain the distinct non-zero integral solutions to (1) as

$$x=10^2[30A^3-690A^2B-1710AB^2+4370]-1$$
$$y=10^2[4A^3-792A^2B-228AB^2+5016]-1$$
$$z=10^2[A^2+19B^2]$$

Properties:

1) $\frac{1}{10^3}[10y(A,1)-x(A,1)+9)-6P_{A-1}^3+723PR_A\equiv 4579(\bmod 667)$

2) $\frac{14}{10^5}(4(x(A,1)+1)-30(y(A,1)+1)+10^5.133)$ is a nasty number

3) $\frac{1}{10^3}[x(A,1)+10y(A,1)+11]-3CP_A^{14}+1722t_{3,A}\equiv 5453(\bmod 466)$

4) $\frac{1}{10^2}[4Az-y+501599]-t_{1586,A}\equiv 0(\bmod 1095)$

2.3 Method 3:

Instead of (5),35can be written as

$$35=\frac{(11+i\sqrt{19})(11-i\sqrt{19})}{2^2}$$

Proceeding as in method 1 and performing some algebra, we obtain the distinct non-zero integral solutions to (1) as

$$x=2^2[12A^3-24A^2B-684AB^2+152B^3]-1$$
$$y=2^2[10A^3-90A^2B-570AB^2+570B^3]-1$$
$$z=2^2[A^2+19B^2]$$

Properties:

1) $x(A,1) - 24SO_A + t_{194,A} \equiv 607(\mathrm{mod}\,2807)$

2) $x(A,1) - y(A,1) - 4[66PR_A - 90Gno_A - 508]$ is a cubical integer.

3) $z(A,1) - 2t_{18,A} - 12t_{12,A} - 31Gno_A - 31$ is a perfect square

4) $y(1,B) - 1140SO_B + 10t_{458,B} \equiv 39(\mathrm{mod}\,1490)$

Note:

Rewriting (7) as

$$((u+1) + i\sqrt{19}v) = (-4 + i\sqrt{19})(a + i\sqrt{19}b)^3 \left(\frac{9 + i\sqrt{19}}{10}\right)$$

and proceeding as in method 2, the non-zero distinct integral solutions to (1) are given by

$$x = 2^2[-10A^3 - 90A^2B + 570AB^2 + 570B^3] - 1$$
$$y = 2^2[-12A^3 - 24A^2B + 684AB^2 + 152B^3] - 1$$
$$z = 2^2[A^2 + 19B^2]$$

2.4 Method 4:

Equation (7) can be written as

$$((u+1) + i\sqrt{19}v) = \frac{(11 + i\sqrt{19})}{2}(a + i\sqrt{19}b)^3 \left(\frac{9 + i\sqrt{19}}{10}\right)$$

Equating real and imaginary parts on both sides and assuming a=2A, b=2B, we get

$$u + 1 = 2^2[8A^3 - 114A^2B - 456AB^2 + 722B^3]$$
$$v = 2^2[2A^3 + 24A^2B - 114AB^2 - 152B^3]$$

Also (4) implies

$$z = 4(A^2 + 19B^2) \quad (10)$$

Substituting the values of u,v in (2) we obtain the non-zero integral solutions to (1) as

$$x = 40[A^3 - 9A^2B - 57AB^2 + 57B^3] - 1$$
$$y = 8[3A^3 - 69A^2B - 171AB^2 + 437B^3] - 1$$

along with (10).

3. CONCLUSION

In this paper, we have made an attempt to obtain a complete set of non-trivial distinct integral solutions for the non-homogeneous ternary cubic equation. To conclude, one may search for other choices of solutions to the considered cubic equation and further cubic equations with multivariables.

NOTATIONS:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

$$P_n^m = \left[\frac{n(n+1)}{6}\right][(m-2)n + (5-m)]$$

$$S_n = 6n(n-1) + 1$$

$$SO_n = n(2n^2 - 1)$$

$$CS_n = n^2 + (n-1)^2$$

$$GnO_n = 2n - 1$$

$$PR_n = n(n+1)$$

$$CP_a^3 = \frac{n(n^2+1)}{2}$$

$$CP_a^9 = \frac{n(3n^2-1)}{2}$$

$$CP_a^{14} = \frac{n(7n^2-4)}{3}$$

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