e AND n -ENERGIES OF LINEAR SEMIGRAPH

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Abstract
In general, graphs are associated with different types of matrices to study their structural properties. As a result, the study of spectrum of graphs become an important area of research in graph theory. Various types of energies have been discussed according to the type of matrix under consideration. In semigraphs, the adjacency between vertices is defined in many ways. This opens a broad scope to study the different energies of semigraphs. In this paper, two types of adjacency matrices, namely e - adjacency matrix and n -adjacency matrix are considered to study the respective energies of the most fundamental semigraph known as linear semigraph.

Keywords: linear semigraph, e - adjacency matrix, n - adjacency matrix, energy of a graph

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1. INTRODUCTION

Semigraphs are introduced as a generalization of graphs. Instead of two vertices as in the graph, several vertices are joined by an edge in a semigraph. Semigraphs were introduced by E. Sampathkumar[5], and studied their applications in the year 2000. Since then, the study of semigraphs become an important area in the field of graph theory. N. S. Bhave et.al[1] studied characterization of potentially Hamiltonian graph in terms of dual semigraph. C. M. Deshpandeet.al [2] studied matrix representation of semigraphs. V. Swaminathan and S. Gomathi [3] studied the domination in semigraphs. Y. B. Venkatakrishnan [6]discussed bipartite theory of semigraphs. N. Murugesan and D. Narmatha [4] discussed the relationship between various types of adjacencies in semigraphs, and properties of associated graphs of a given semigraph. In this paper, two different energies of semigraphs have been discussed.

2. PREREQUISITIES

A semigraph S is a pair (V, X) where V is a nonempty set whose elements are called vertices of S and X is a set of ordered n-tuples $n \geq 2$, called edges of S satisfying the following conditions:
i. The components of an edge E in X are distinct vertices from V .
ii. Any two edges have at most one vertex in common.
iii.   Two   edges $E_1 = (u_1, u_2, \ldots\ldots u_m)$ and
$E_2 = (v_1, v_2, \ldots. v_n)$ are said to be equal iff
     a. $m = n$ and
     b. either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$.

The vertices in a semigraph are divided into three types namely end vertices, middle vertices and middle-end vertices, depending upon their positions in an edge. The end vertices are represented by thick dots, middle vertices are represented by small circles, a small tangent is drawn at the small circles to represent middle-end vertices.

There are different types of adjacency of two vertices in a semigraph. Two vertices u and v in a semigraph S are said to be
i. adjacent if they belong to the same edge.
ii. consecutively adjacent if they are adjacent and consecutive in order as well.
iii. e-adjacent if they are the end vertices of an edge.
iv. 1e-adjacent if both the vertices u and v belong to the same edge and atleast one of them is an end vertex of that edge.
v. $n-$ adjacent if they belong to n edges.

Correspondingly different types of adjacent matrices are defined, and they are used to study different kinds of structural properties of the semigraph. The natural construction of a semigraph paves a way to introduce various types of adjacencies among the vertices in a semigraph, and hence various adjacent matrices exist. The sum of eigenvalues of a particular adjacent matrix is called the respective energy of the graph in general. The same can also be applied to semigraphs. Since the structural components i.e., the vertices and edges and their complexicities interms connections make the task of studying the energy of semigraph is difficult as compared to the general graphs. As a prelude to this endeavour , we consider the most fundamental semigraph and two types of adjacent matrices to discuss the corresponding energy to this particular type of semigraphs.

3. e – ADJACENCY   MATRIX   OF   A   SEMIGRAPH

3.1 Definition

Let $S = (V, X)$ be   a   semigraph   with   vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $X = \{e_1, e_2, ..., e_q\}$ where $e_j = \left(i_1, i_2, ...i_{kj}\right), j = 1, 2, ..., q$ and $i_1, i_2, ...i_{kj}$ are distinct elements of V.

The e – adjacency matrix A of $S = (V, X)$ is defined as

$A = (a_{ij})_{nxn}$ so   that

$a_{ij} = \begin{cases} 1, & if \ v_i \ and \ v_j \ are \ end \ vertices \ of \ e_k, 1 \le k \le q \\ 0, & otherwise \end{cases}$

3.2 Linear Semigraph

A semigraph with a single edge and n vertices is called a linear semigraph L_n.

3.3 e – Eigen values of a Linear Semigraph

Consider a semigraph with single edge having two end vertices and one middle vertex, then its e – adjacency matrix A_3 is as follows:

$A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and

$|A_3 - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(\lambda^2) + \lambda = -\lambda^3 + \lambda$

i.e., $\varphi(L_3, \lambda) = -\lambda^3 + \lambda$.

For the semigraph with single edge having two end vertices and two middle vertices, then its e – adjacency matrix is

$A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and

$|A_4 - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = \lambda^4 - \lambda^2$

The following table gives the coefficients of $\varphi(L_n, \lambda)$ for $n = 2, 3, ..., 9$.

Table 1

n	b_9	b_8	b_7	b_6	b_5	b_4	b_3	b_2	b_1	b_0
2	-	-	-	-	-	-	-	1	0	-1
3	-	-	-	-	-	-	1	0	-1	0
4	-	-	-	-	-	1	0	-1	0	0
5	-	-	-	-	1	0	-1	0	0	0
6	-	-	-	1	0	-1	0	0	0	0
7	-	-	1	0	-1	0	0	0	0	0
8	-	1	0	-1	0	0	0	0	0	0
9	1	0	-1	0	0	0	0	0	0	0

From the above table it can be generalized that

$$\varphi(L_n, \lambda) = (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-2} = (-1)^{n-1} \lambda^{n-2}[1 - \lambda^2]$$

Also, we have $\varphi(L_{n-1}, \lambda) = (-1)^{n-2} \lambda^{n-3}[1 - \lambda^3]$

Therefore, $\varphi(L_n, \lambda) = (-\lambda)\varphi(L_{n-1}, \lambda)$

The following proposition gives the properties of the characteristic polynomial of the e – adjacency matrix of a linear semigraph.

3.4 Proposition

The characteristic polynomial $\varphi(L_n, \lambda)$ of a linear semigraph L_n satisfy the following properties.

i. $\varphi(L_n, \lambda) = \beta_0 \lambda^n + \beta_1 \lambda^{n-1} + \beta_2 \lambda^{n-2} + ... + \beta_n$

where $\beta_0 = (-1)^n; \beta_2 = (-1)^{n-1}$ and $\beta_1 = \beta_3 = ... = \beta_n = 0$

ii. $\varphi(L_n, \lambda) = (-1)^{n-2} \lambda^{n-2} \varphi(L_2, \lambda)$ for $n \ge 2$ with $\varphi(L_2, \lambda) = \lambda^2 - 1$.

Proof
We use mathematical induction to prove the proposition. When $n = 2$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\varphi(L_2, \lambda) = |A - \lambda I| = \lambda^2 - 1.$$

i.   e., $\varphi(L_2, \lambda) = \beta_0 \lambda^2 + \beta_1 \lambda + \beta_2$ where $\beta_0 = 1; \beta_1 = 0; \beta_2 = -1$.

Hence (i) is true, when $n = 2$.

As induction hypothesis, assume that

$$\varphi(L_{n-1}, \lambda) = \beta_0 \lambda^{n-1} + \beta_1 \lambda^{n-2} + \beta_2 \lambda^{n-3} + ... + \beta_{n-1}$$

\where $\beta_0 = (-1)^{n-1}; \beta_2 = (-1)^{n-2}$ and $\beta_1 = \beta_3 = ... = \beta_{n-1} = 0$.

$$\varphi(L_n, \lambda) = -\lambda \varphi(L_{n-1}, \lambda)$$

$$= -\lambda \left[\beta_0 \lambda^{n-1} + \beta_1 \lambda^{n-2} + \beta_2 \lambda^{n-3} + ... + \beta_{n-1} \right]$$

$$= -\lambda \left[(-1)^{n-1}.\lambda^{n-1} + (-1)^{n-2} \lambda^{n-3} \right]$$

$$= (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-2}$$

$$= \beta_0 \lambda^n + \beta_1 \lambda^{n-1} + \beta_2 \lambda^{n-2} + ... + \beta_n$$

where $\beta_0 = (-1)^n; \beta_2 = (-1)^{n-1}$ and $\beta_1 = \beta_3 = ... = \beta_n = 0$.

Hence, by induction principle (i) is true for all values of n.

Similarly we prove (ii) also by mathematical induction. It can be easily seen that (ii) is true when $n = 2$. Hence, assume that

$$\varphi(L_{n-1}, \lambda) = (-1)^{n-3} \lambda^{n-3} \varphi(L_2, \lambda)$$

We have

$$\varphi(L_n, \lambda) = (-\lambda) \varphi(L_{n-1}, \lambda)$$
$$= (-\lambda)(-1)^{n-3} \lambda^{n-3} \varphi(L_2, \lambda) = (-1)^{n-2} \lambda^{n-2} \varphi(L_2, \lambda).$$

Thus (ii) is true for all values of n.

3.5 e – Energy of a Linear Semigraph

L_n be a linear semigraph with n vertices. If A is the e – adjacency matrix of L_n, then the eigen values of A, denoted by $\lambda_1, \lambda_2, ..., \lambda_n$, are said to be the eigen values of L_n.

The e – energy of the linear semigraph L_n is defined as

$$E(L_n) = \sum_{i=1}^{n} |\lambda_i|$$

From proposition 3.4

$$\varphi(L_n, \lambda) = (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-2}$$

When n = 2,

The eigen values of $\varphi(L_2, \lambda) = \lambda^2 - 1$ are 1,-1 and the e – energy of $L_2 = 2$.

The eigen values of $\varphi(L_3, \lambda) = -\lambda^3 + \lambda$ are 1,-1,0 and the e – energy of $L_3 = 2$.

Similarly   the   eigen   values   of $\varphi(L_n, \lambda) = (-1)^n \lambda^n + (-1)^{n-1} \lambda^{n-2}$ are 1,-1,0(n-2 times).

For example, consider a linear semigraph with $n = 5$, its e – adjacency matrix is

$$A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of L_5 is $\lambda^5 - \lambda^3 = 0$ and the eigen values are 0,0,0,-1,1.

i.e., $E(L_5) = 2$. On generalization, we obtain the following theorem.

3.6 Theorem

The e – energy of a linear semigraph L_n is the number of end vertices in L_n.

4. n – ADJACENCY MATRIX OF A LINEAR SEMIGRAPH

4.1 Definition

The $n-$adjacency matrix A of $L = (V, X)$ is defined as

$A = (a_{ij})_{nxn}$ so
that
$$a_{ij} = \begin{cases} n, & if\ there\ are\ E_1, E_2, ..., E_n\ containing\ v_i, v_j \\ 0, & otherwise \end{cases}$$

4.2 $n-$Eigen values of a Linear Semigraph

The $n-$adjacency matrix of A_3 is

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

then $\varphi(L_3, \lambda) = -\lambda^3 + 3\lambda^2$.

Similarly $\varphi(L_4, \lambda) = \lambda^4 - 4\lambda^3$

The following table gives the coefficients of $\varphi(L_n, \lambda)$ for $n = 2, 3, ..., 9$.

Table 2

n	b_9	b_8	b_7	b_6	b_5	b_4	b_3	b_2	b_1
2	-	-	-	-	-	-	-	1	-2
3	-	-	-	-	-	-	-1	3	0
4	-	-	-	-	-	1	-4	0	0
5	-	-	-	-	-1	5	0	0	0
6	-	-	-	1	-6	0	0	0	0
7	-	-	-1	7	0	0	0	0	0
8	-	1	-8	0	0	0	0	0	0
9	-1	9	0	0	0	0	0	0	0

From the above table it can be generalized that

$$\varphi(L_n, \lambda) = (-1)^n \lambda^n + n(-1)^{n-1}\lambda^{n-1} = (-1)^{n-1}\lambda^{n-1}[n - \lambda]$$

Also, we have

$$\varphi(L_{n-1}, \lambda) = (-1)^{n-2}\lambda^{n-2}[(n-1) - \lambda]$$

Therefore, $\varphi(L_n, \lambda) = (-\lambda)\varphi(L_{n-1}, \lambda) + (-\lambda)^{n-1}$

The following proposition gives the properties of the characteristic polynomial of the $n-$adjacency matrix of a linear semigraph.

4.3 Proposition

The characteristic polynomial $\varphi(L_n, \lambda)$ of a linear semigraph L_n satisfy the following property.

$$\varphi(L_n, \lambda) = \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + ... + \alpha_n$$

where $\alpha_0 = (-1)^n; \alpha_1 = n(-1)^{n-1}$ and
$\alpha_2 = \alpha_3 = ... = \alpha_n = 0$

Proof

We use mathematical induction to prove the proposition. When $n = 2$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\varphi(L_2, \lambda) = |A - \lambda I| = \lambda^2 - 2\lambda.$$

Ie., $\varphi(L_2, \lambda) = \alpha_0\lambda^2 + \alpha_1\lambda + \alpha_2$ where
$\alpha_0 = 1; \alpha_1 = -2; \alpha_2 = 0$.

Hence the proposition is true, when $n = 2$.

As induction hypothesis, assume that

$$\varphi(L_{n-1}, \lambda) = \alpha_0\lambda^{n-1} + \alpha_1\lambda^{n-2} + \alpha_2\lambda^{n-3} + ... + \alpha_{n-1}$$

\where $\alpha_0 = (-1)^{n-1}; \alpha_1 = (n-1)(-1)^{n-2}$ and

$$\alpha_2 = \alpha_3 = ... = \alpha_{n-1} = 0.$$

$$\varphi(L_n, \lambda) = -\lambda\varphi(L_{n-1}, \lambda) + (-\lambda)^{n-1}$$

$$= -\lambda[\alpha_0\lambda^{n-1} + \alpha_1\lambda^{n-2} + \alpha_2\lambda^{n-3} + ... + \alpha_{n-1}] + (-\lambda)^{n-1}$$

$$= -\lambda[(-1)^{n-1}\lambda^{n-1} + (n-1)(-1)^{n-2}\lambda^{n-2}] + (-\lambda)^{n-1}$$

$$= (-1)^n\lambda^n + n(-1)^{n-1}\lambda^{n-1}$$

$$= \alpha_0\lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + ... + \alpha_n$$

where $\alpha_0 = (-1)^n; \alpha_1 = n(-1)^{n-1}$ and
$\alpha_2 = \alpha_3 = ... = \alpha_n = 0$.

Hence, by induction principle the proposition is true for all values of n.

4.4 n – Energy of a Linear Semigraph

From proposition 4.3

$$\varphi(L_n, \lambda) = (-1)^n \lambda^n + n(-1)^{n-1} \lambda^{n-1}$$

The eigen values of $\varphi(L_2, \lambda) = \lambda^2 - 2\lambda$ are 0,2 and the energy of $L_2 = 2$.

The eigen values of $\varphi(L_3, \lambda) = -\lambda^3 + 3\lambda^2$ are 0,0,3 and the energy of $L_3 = 3$.

Similarly the eigen values of $\varphi(L_n, \lambda) = (-1)^n \lambda^n + n(-1)^{n-1} \lambda^{n-1}$ are 0(n-1 times), n.

i.e., $E(L_n) = n$. On generalization, we get the following theorem.

4.5 Theorem

The energy of a linear semigraph L_n is the number of vertices in L_n.

5. CONCLUSION

In this paper, e – adjacency matrix and n – adjacency matrix are defined and the respective energies have been discussed considering the most fundamental semigraph. This work can be generalized to an arbitrary semigraph.

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