INTEGRATION TECHNIQUES FOR TWO DIMENSIONAL DOMAINS

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Abstract

In this work, three different integration techniques, which are the numerical, semi-analytical and exact integration techniques are briefly reviewed. Numerical integrations are carried out using three different Quadrature rules, which are the Classical Gauss Quadrature, Gauss Legendre and Generalized Gaussian Quadrature. Line integral method is used to perform semi-analytical integration, while the generalized equations developed by the author in previous works are used to carry out the exact integration. It is seen that Generalized Gaussian Quadrature rules outperforms other Quadrature rules during the numerical and semi-analytical integrations. It is also shown that the exact integration technique developed by the author in previous works yield accurate results for integration of monomials.

Keywords: *Numerical Integration, Semi-Analytical Integration, Exact Integration, Quadrature Rules, Classical Gauss Quadrature, Gauss Legendre, Generalized Gaussian Quadrature, Line Integral, and Generalized Equations*

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1. INTRODUCTION

Integration of a function over a certain domain is frequently encountered in engineering computations. Finite Element Method (FEM) is one of the computational methods which require such integration technique to obtain the stiffness matrix, k according to the formula below (for solid mechanics):

$$k = \int_{\Omega} B^T D \ B \ \ d\Omega \qquad (1)$$

Where Ω represents domain, B represents strain displacement matrix and D represents matrix property matrix. The stiffness matrix, k is later used to determine response of a material to external loads, such as the nodal displacements, strain and stress components. Apart from solid mechanics, similar integration is also encountered in other engineering applications (which are analyzed using FEM) such as in thermal problems, fluid flow problems and etc. Thus, there have been many integration techniques developed and proposed to perform the integration. Most common technique used in FEM to perform the integration is by using numerical integration technique. Numerical integration is preferred, since it yields good results at lower computational time compared to analytical method. Analytical solutions give accurate results, but require high computational time. Furthermore, analytical solutions might not exist for certain cases.

Numerical integration is performed by using quadrature points and weights, which are generated specifically for certain functions and/or domain. Three different quadrature rules are investigated in this work, namely Classical Gauss Quadrature, Gauss Legendre and Generalized Gaussian Quadrature. The integration points and weights are obtained by solving a set of functions within certain interval. These set of functions could be polynomials, trigonometric functions, and basis functions of a particular function space which define the domain **[1]**. Resulting Quadrature rules (integration points and weights) can then be used to solve integrands of similar type with the set of functions chosen earlier. Gaussian Quadrature methods use a set of polynomial functions to evaluate the integration points and weights, and thus it can be used to integrate polynomial functions accurately, but performs poorly when the function to be integrated (integrand) is different from polynomials such as functions with fractional power, trigonometric functions and etc. In order to solve for different type of integrands, a new set of functions of similar type as the integrands to be solved for need to be selected and solved for new Quadrature rules. Authors in **[2]** have proposed to replace the set of functions used to solve for the Quadrature rules from polynomials to functions from wider classes, in order to be able to solve integrands from various function types. Based on this concept, authors in **[3]** have successfully presented a numerical scheme for Generalized Gaussian Quadrature which can be used to integrate any variety of functions, including smooth functions and functions with singular end points.

Recent advances in numerical integration can be found within the context of polygonal finite element method (PFEM). Those techniques are **[4]**:

- Partitioning of the polygonal domains into several triangles and later perform integration using numerical quadrature rules onto these triangles.
- Partitioning the master element into several triangles and perform integration using numerical quadrature rules onto these triangles in the master element with isoparametric mapping
- By utilizing cubature rules for irregular polygons based on triangles or conforming mapping.
- By utilizing generalized quadrature rules on triangles or polygons based on symmetry groups and numerical optimization.

Several techniques to generate new higher order quadrature rules for triangles can also be seen in **[5, 6]**.

Semi-analytical integration was later introduced, to increase the accuracy of the solution. Author in **[7]** has proposed a new semi-analytical integration technique which reduces double integration to single integration through use of divergence theorem. An integrand is integrated analytically with respect to variable *x* first, and next integration with respect to variable *y* is carried out numerically, using Quadrature rule of choice.

Finally, exact integration techniques were proposed to improve solution accuracy. Such technique has been developed for quadrilateral elements in FEM. In [**8**], the author proposed bilinear mapping to transform distorted quadrilaterals to the unit square and later integration is performed using matrix convolution. Another development of exact integration technique over quadrilaterals can be seen in **[9]**. The author developed an exact integration technique for quadrilaterals which are formulated using Wachspress shape functions. Exact integration formulas for strain based quadrilateral elements were later developed in [**10**].

This paper is arranged as follows. Section 2 covers mathematical background for the Quadrature rules used in this paper. Numerical integration, semi-analytical integration and exact integration techniques are explained in sections 3, 4 and 5, respectively. Comparison between all these integration techniques are done in section 6 through numerical simulations and the paper is finally concluded in section 7.

2. MATHEMATICAL BACKGROUND

Mathematical backgrounds for the three different Quadrature rules are provided in the following sub-headings.

2.1 Gauss Legendre

In case of Gauss Legendre ($w(x)$ = 1), the set of functions used to solve for integration points and weights are Legendre polynomials **[11]**.

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n\, m!(n-m)!(n-2m)!} x^{n-2m}$$
$$M = n/2 \quad or \qquad (2)$$
$$M = {(n-1)}/{2}, \ whichever \ is \ an \ integer$$

Where *n* represents the polynomial degree. The integration points are the roots for the Legendre polynomials and can be determined using numerical technique such as Newton-Raphson;

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
$$with \ initial \ guess \ x_0 = \cos\left(\frac{r - 1/4}{n + 1/2}\right)\pi \qquad (3)$$

Where *i* represents the iteration number, *n* represents the polynomial degree, and *r* represents the r^{th} root of the polynomial. Once integration points x_i are determined, the weights can be calculated using the relation:

$$w_i = \frac{2\left(1 - x_i^2\right)}{(n+1)^2\left[P_{n+1}(x_i)\right]^2} \qquad (4)$$

Integration interval for Gauss Legendre is [-1, 1].

2.2 5th Order Classical Gaussian Quadrature

Classical Gauss Quadrature ($w(x)$ = 1) integrates general polynomials with order 2*n*-1 exactly, where *n* represents integration order. The integration interval is [0, 1]. The general polynomials are of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n-1} x^{2n-1} \qquad (5)$$

Thus, the integration can be represented as:

$$\int_a^b f(x)\,dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n-1} x^{2n-1}\right) dx$$
$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + \ldots \qquad (6)$$
$$\ldots + a_{2n-1}\left(\frac{b^{2n} - a^{2n}}{2n}\right)$$

Rewriting the integration in numerical form and substituting the function with general polynomials (equation (5)) gives;

$$\int_a^b f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + \ldots + w_n f(x_n)$$
$$= w_1\left(a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_{2n-1} x_1^{2n-1}\right) + \ldots \qquad (7)$$
$$\ldots + w_2\left(a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_{2n-1} x_2^{2n-1}\right) + \ldots$$
$$\ldots + w_n\left(a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_{2n-1} x_n^{2n-1}\right)$$

Integration points x_i and weights w_i can then be obtained by comparing equations (6) and (7) to get a list of 2*n* equations consisting of variables x_i , w_i , *a* and *b* and solve them simultaneously. Integration limits *a* and *b* depend on Quadrature rule of choice. For 5th order Classical Gauss Quadrature, *n* = 5, *a* = 0 and *b* = 1. Thus, equations (6) and (7) become:

$$\int_a^b f(x)\,dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_9 x^9\right)dx$$
$$= a_0 + a_1\left(\frac{1}{2}\right) + a_2\left(\frac{1}{3}\right) + \ldots + a_9\left(\frac{1}{10}\right) \tag{8}$$

$$\int_a^b f(x)\,dx = w_1 f(x_1) + w_2 f(x_2) + \ldots + w_n f(x_n)$$
$$= w_1\left(a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_9 x_1^9\right) + \ldots$$
$$\ldots + w_2\left(a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_9 x_2^9\right) + \ldots$$
$$\ldots + w_5\left(a_0 + a_1 x_5 + a_2 x_5^2 + \ldots + a_9 x_5^9\right) \tag{9}$$

2.3 Generalized Gaussian Quadrature

Gaussian Quadrature method such as Gauss Legendre uses set of polynomial functions to evaluate the integration points and weights, and thus the method performs well when the integrand is a polynomial function. The method performs poorly when the integrand is different from polynomial function.

J. Ma, V. Rokhlin and S. Wandzura **[3]** replaced the set of functions used to solve for the Quadrature rules from polynomials to functions from wider class and successfully presented a numerical scheme for Generalized Gaussian Quadrature. Number of functions used for generating the Quadrature rules is 2*n*, where *n* represents integration order. The Quadrature rules presented in **[3]** can be used to integrate any variety of functions, including smooth functions and functions with singular end points. The integration interval is [0, 1].

3. NUMERICAL INTEGRATION

Integration over a 2-dimensional domain can be represented using Fubini's theorem **[12]**:

$$I = \begin{cases} \displaystyle\int_a^b \int_{r(x)}^{s(x)} f(x,y)\,dy\,dx \\ \quad or \\ \displaystyle\int_a^b \int_{r(y)}^{s(y)} f(x,y)\,dx\,dy \end{cases} \tag{10}$$

Domain to be integrated can be enclosed by:

- 4 constant lines
- 3 constant lines and 1 function
- 2 constant lines and 2 functions

It can be seen in the previous sub-headings **2.1-2.3** that each Quadrature rule is developed based on different intervals. Thus, the integration limits *a, b, r* and *s* in the **equation (10)** above need to be converted according to the Quadrature rule of choice. This can be achieved by utilizing the generalized equations which have been developed by Logah et all **[13, 14]**:

$$\left.\begin{array}{l} I_1 = \displaystyle\int_a^b \int_{r(x)}^{s(x)} f(x,y)\,dy\,dx \\ \quad or \\ I_2 = \displaystyle\int_a^b \int_{r(y)}^{s(y)} f(x,y)\,dx\,dy \end{array}\right\} = \int_L^U \int_L^U f(m_x u + c_x, m_y v + c_y)\, m_x\, m_y\, dv\, du \tag{11}$$
$$= \sum_j^n \sum_i^n w_i w_j m_x\, m_y\; f(m_x u_i + c_x, m_y v_i + c_y)$$

The *U* and *L* in the **equation (11)** above represents upper and lower limits of the integration, which can be set as 0, 1, or -1, depending on the Quadrature rule of choice.

4. SEMI-ANALYTICAL INTEGRATION

Line integral which was introduced by G. Dasgupta, **[7]** enables integration to be carried out without a need to partition the region. The method reduces double integration to single integration through use of divergence theorem. An integrand is integrated analytically with respect to variable *x* first, and next integration with respect to variable *y* is carried out using one dimensional Quadrature rule. Formula for integration of a function with respect to the second variable is recalled from **[7]**:

$$(y_2 - y_1)\int_0^1 f[x_1 + \tau(x_2 - x_1), y_1 + \tau(y_2 - y_1)]\,d\tau \tag{12}$$

Single integration given by equation (12) is carried out numerically using Classical Gauss Legendre, Generalized Gaussian Quadrature and 5th order Gaussian Quadrature. Equation (12) is converted to numerical form through the following relation:

$$I = \int_a^b f(x)w(x)dx = \sum_{i=1}^n w_i\, f(x_i) \tag{13}$$

where *a*, *b* represents integration limits, *f(x)* represents function to be integrated (integrand), *w(x)* represents weight functions, w_i represents integration weights, xi represents integration points, *i* = 1,2,3,…*n*, and *n* represents integration order.

5. EXACT INTEGRATION

The generalized **equation (11)** converts arbitrary integration limits of *a*, *b*, *r* and *s* to specified interval [*U*, *L*] without involving any symbolic computation (fully numerical). In order to derive expressions for exact integration of monomials, the integration limits (or the functions covering the domain) should be converted to [0, 1]. Analytical integration of monomials $x^m y^n$ can then be represented numerically using the relation **[14]**:

$$\left.\begin{array}{c}\displaystyle\int_0^1\int_0^1 x^m y^n \, dy\,dx \\ or \\ \displaystyle\int_0^1\int_0^1 x^m y^n \, dx\,dy\end{array}\right\} = \frac{1}{(m+1)(n+1)} \qquad (14)$$

It can be seen that the method shown in **equation (14)** requires handling of symbols to identify the variables *x, y* and their respective powers, *m* and *n*. The exact integration method is limited to polynomials functions.

6. NUMERICAL EXAMPLE

Set of functions *f* (*x*, *y*) are integrated using the integration schemes above (numerical, semi-analytical and exact integration techniques), over a symmetrical hexagon with coordinates shown in **Fig -1(a)**. Set of functions selected include polynomial function, rational function, function with rational power, natural logarithmic function and exponential function. The symmetrical hexagon is separated into 2 regions; R_1 and R_2 according to the requirement of Fubini's Theorem, as shown in **Fig -1(b)**. Each region is enclosed by two constant lines and two linear functions.

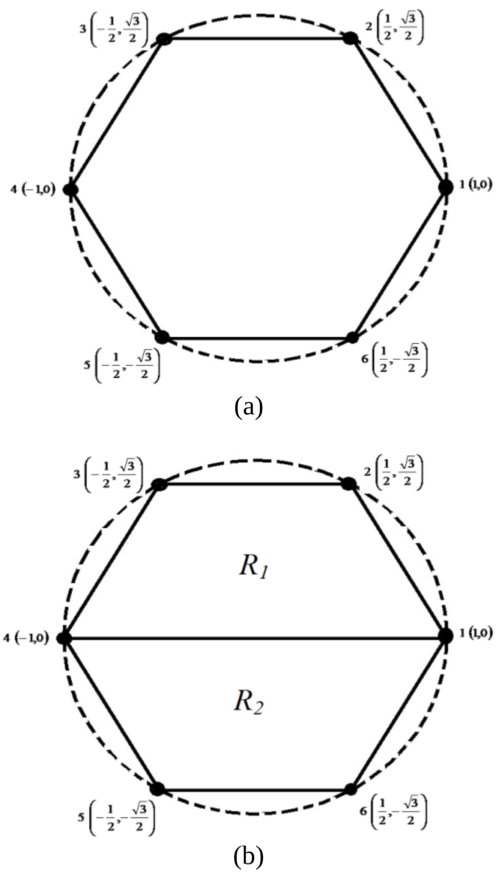

Fig -1: Example of domain with linear sides in two dimensions (a) Symmetrical hexagonal element without partitioning (b) Partitioned hexagonal element

Integration of a function over the entire domain is then given by:

$$I = \iint_{R_1} f(x,y)\,dxdy + \iint_{R_2} f(x,y)\,dxdy$$
$$I = \int_0^{\frac{\sqrt{3}}{2}} \int_{\left(\frac{y}{\sqrt{3}}-1\right)}^{\left(1-\frac{y}{\sqrt{3}}\right)} f(x,y)\,dxdy + \int_{-\frac{\sqrt{3}}{2}}^{0} \int_{\left(-1-\frac{y}{\sqrt{3}}\right)}^{\left(\frac{y}{\sqrt{3}}+1\right)} f(x,y)\,dxdy \qquad (15)$$

The semi-analytical technique is performed over the domain boundary and does not require partitioning of the domain, while the exact integration technique is limited to polynomial functions. Results for the integration techniques are as shown in **Tables 1-6**. Percentage error is calculated based on **equation (16)**:

$$\%\,Error = \frac{Analytical\ solution - calculated\ solution}{Analytical\ solution} \times 100\% \qquad (16)$$

From the results in the **Tables 1-6**, it is seen that for numerical integration technique, the Generalized Gaussian Quadrature tends to provide converging solution when integration order is increased. For all the functions which have been tested, Generalized Gaussian Quadrature outperforms Classical Gauss Legendre and 5th order Gaussian Quadrature. Fractional functions could not be solved using Classical Gauss Legendre and 5th order Gaussian Quadrature for certain integration orders. This is due to the fact that some of the integration points or weights of these rules contain value of zero (leads to singular point). Apart from that, these Quadrature rules were not generated based on fractional functions.

As for semi-analytical integration technique, the Classical Gauss Legendre and 5th order Gaussian Quadrature are able to solve fractional functions. This is caused by the analytical integration of the first variable. Accuracy of the results obtained from this technique has slightly improved compared to the numerical integration technique. However, this technique would require more computational time compared to the numerical integration technique, due to the symbolic manipulation.

Finally, the exact integration technique yields accurate solutions at lower computational time compared to the analytical solutions. However, this technique is applicable for monomials only. The exact integration technique is suitable for FEM based on stiffness matrices which consist of simple polynomials, such as virtual node method and strain based elements. This technique provides an alternative for existing numerical integration techniques, especially Gaussian quadrature which requires high number of integration points and weights to integrate higher order monomials.

7. CONCLUSIONS

Accuracy of three different Quadrature rules - Classical Gauss Quadrature, Gauss Legendre and Generalized Gaussian Quadrature in performing numerical integration within two dimensional domains have been successfully compared through simulations. Three different integration techniques have been demonstrated in this work, which are the numerical, semi-analytical and exact integration techniques. The integration limits were converted accordingly through utilization of the generalized equations developed by the author in previous work **[13, 14].** It is seen that Generalized Gaussian Quadrature performs better compared to other Quadrature rules. Semi-analytical method is recommended for simple integrals and when high accuracy is required, since the method requires high computational time and solution might not exist for complex integrals. The exact integration technique is applicable for FEM based on stiffness matrices which consist of simple polynomials, such as virtual node method and strain based elements.

Table -1: Analytical solutions for integration of various functions over the symmetrical hexagonal domain.

Function $f(x, y)$	Analytical solution
x^2+y	$\frac{5\sqrt{3}}{16}$
$1 + (y/x)$	$\frac{3\sqrt{3}}{2}$
$(1+x)^{1/2}$	$\left(\frac{1}{15}\sqrt{2}\right)\times\left(-27+31\sqrt{3}\right)$
$(x+y)(\ln y)$	$-\frac{\pi}{2}i$
e^{1+x}	$2\sqrt{3}\times\left(-1+\sqrt{e}\right)^2\times(1+\sqrt{e}+e)$
x^2+2y^4	$-\frac{46\sqrt{3}}{80}$
$3x^3y^4+2x^2y^3$	$-\frac{6}{70}$

Table -2: Numerical integration technique - Results obtained for various Quadrature rules and integration orders, *n*.

Function $f(x, y)$	n	Classical Gauss Legendre	Generalized Gaussian Quadrature	5^{th} order Gaussian Quadrature as used by G. Dasgupta, **[7]**
x^2+y	5	0.5412658773652735	0.5412658773652659	0.5412639521009471
	10	0.5412658773918141	0.5412658773652766	
	15	0.5412658773652723	0.541265877365275	
	20	0.5412658773090507	0.5412658773652741	
$1 + (y/x)$	5	Infinity (Division by zero)	2.5980762113532894	Infinity (Division by zero)
	10	2.598076211353316	2.598076211353329	
	15	Infinity (Division by zero)	2.5980762113532236	
	20	2.598076211352796	2.5980762113533147	
$(1+x)^{1/2}$	5	2.5168891296282347	2.5167180435246514	2.5168790706365485
	10	2.5167025897573976	2.5166950838508293	
	15	2.5166955774486093	2.516694489735185	
	20	2.516694684756974	2.5166944145358086	
$(x+y)(\ln y)$	5	-1.5707963267948952 i	-1.5707963267948541 i	−1.5707898672794112 i
	10	-1.5707963267861438 i	-1.570796326794901 i	
	15	-1.5707963267948912 i	-1.5707963267948981 i	
	20	-1.5707963268130878 i	-1.5707963267948954 i	
e^{1+x}	5	7.824178417294142	7.824165889257099	7.824147591048949
	10	7.824178417977526	7.824178417939303	
	15	7.824178417939274	7.824178417939303	
	20	7.824178417856461	7.824178417939299	

Table -3: Numerical integration technique - Percentage error

Function *f (x, y)*	*n*	Classical Gauss Legendre	Generalized Gaussian Quadrature	5^{th} order Gaussian Quadrature as used by G. Dasgupta, **[7]**
x^2+y	5	1.02558×10^{-13}	1.51786×10^{-12}	3.55697×10^{-4}
	10	-4.90332×10^{-9}	-4.71767×10^{-13}	
	15	3.28186×10^{-13}	-1.64093×10^{-13}	
	20	1.03874×10^{-8}	0	
1 + (y / x)	5	N/A	1.02558×10^{-12}	N/A
	10	0	-4.95697×10^{-13}	
	15	N/A	3.55534×10^{-12}	
	20	2.00159×10^{-11}	5.1279×10^{-14}	
$(1+x)^{1/2}$	5	-7.73793×10^{-3}	-9.3988545×10^{-4}	-7.33824×10^{-3}
	10	-3.25835×10^{-4}	-2.75906×10^{-5}	
	15	-4.72035×10^{-5}	-3.98359×10^{-6}	
	20	-1.17327×10^{-5}	-9.95571×10^{-7}	
(x+y)(ln y)	5	8.48148×10^{-14}	2.69994×10^{-12}	4.11226×10^{-4}
	10	5.57219×10^{-10}	-2.82716×10^{-13}	
	15	3.39259×10^{-13}	-9.89506×10^{-14}	
	20	-1.15809×10^{-9}	7.0679×10^{-14}	
e^{1+x}	5	8.24569×10^{-9}	1.60128×10^{-4}	3.93995×10^{-4}
	10	-4.88566×10^{-10}	-3.40551×10^{-14}	
	15	3.292×10^{-13}	-3.40551×10^{-14}	
	20	1.05875×10^{-9}	1.13517×10^{-14}	

Table -4: Semi-analytical integration technique - Results obtained for various Quadrature rules and integration orders, *n*.

Function *f (x, y)*	*n*	Classical Gauss Legendre	Generalized Gaussian Quadrature	5^{th} order Gaussian Quadrature as used by G. Dasgupta, **[7]**
x^2+y	5	0.5412658773652739	0.5412658773652702	0.5412648150178867
	10	0.5412658773676868	0.5412658773652751	
	15	0.5412658773652732	0.5412658773652742	
	20	0.5412658773601184	0.541265877365274	
1 + (y / x)	5	2.5980762113533147	2.5980800823789885	2.598070986673093
	10	2.5980762113533156	2.5980762113637423	
	15	2.598076211353312	2.598076211353317	
	20	2.5980762113530558	2.598076211353316	
$(1+x)^{1/2}$	5	2.516701554669756	2.5167125922506286	2.5166965172778113
	10	2.5166946579427094	2.51669508551611	
	15	2.516694427387341	2.5166944898637604	
	20	2.5166943988138315	2.5166944145272927	
(x+y)(ln y)	5	-1.5707963267948957 i	-1.5707963267948821 i	-1.5707930878087952 i
	10	-1.570796326786144 i	-1.570796326794899 i	
	15	-1.5707963267948941 i	-1.5707963267948966 i	
	20	-1.5707963268132448 i	-1.5707963267948968 i	
e^{1+x}	5	7.824178417939292	7.824178412557625	7.824162799649706
	10	7.82417841794291	7.824178417939315	
	15	7.824178417939288	7.824178417939303	
	20	7.824178417930874	7.8241784179393	

Table -5: Semi-analytical integration technique - Percentage error

Function $f(x, y)$	*n*	Classical Gauss Legendre	Generalized Gaussian Quadrature	5^{th} order Gaussian Quadrature as used by G. Dasgupta, **[7]**
x^2+y	5	4.10232×10^{-14}	7.17906×10^{-13}	1.96271×10^{-4}
	10	-4.45758×10^{-10}	-1.84604×10^{-13}	
	15	1.64093×10^{-13}	-2.05116×10^{-14}	
	20	9.52518×10^{-10}	2.05116×10^{-14}	
$1 + (y/x)$	5	5.1279×10^{-14}	-1.48996×10^{-4}	2.01098×10^{-4}
	10	1.7093×10^{-14}	-4.01309×10^{-10}	
	15	1.53837×10^{-13}	-3.4186×10^{-14}	
	20	1.00165×10^{-11}	0	
$(1+x)^{1/2}$	5	-2.84706×10^{-4}	-7.23281×10^{-4}	-8.45473×10^{-5}
	10	-1.06673×10^{-5}	-2.76567×10^{-5}	
	15	-1.50622×10^{-6}	-3.9887×10^{-6}	
	20	-3.70864×10^{-7}	-9.95233×10^{-7}	
$(x+y)(\ln y)$	5	5.65432×10^{-14}	9.18827×10^{-13}	2.062×10^{-4}
	10	5.57205×10^{-10}	-1.55494×10^{-13}	
	15	1.55494×10^{-13}	0	
	20	-1.16808×10^{-9}	-1.41358×10^{-14}	
e^{1+x}	5	1.02165×10^{-13}	6.87826×10^{-8}	1.99616×10^{-4}
	10	-4.61447×10^{-11}	-1.92979×10^{-13}	
	15	1.47572×10^{-13}	-3.40551×10^{-14}	
	20	1.07694×10^{-10}	0	

Table -6: Exact integration technique - Results obtained for polynomial functions.

Function $f(x, y)$	Solution from Exact Integration Technique	Percentage error (%)	Average maximum time elapsed for exact integration technique (seconds)	Average maximum time elapsed for analytical technique (seconds)
x^2+2y^4	$R_1 = -\frac{23\sqrt{3}}{80}$ $R_2 = -\frac{23\sqrt{3}}{80}$	0	0.13 for R_1 0.13 for R_2	0.44 for R_1 0.45 for R_2
$3x^3y^4+2x^2y^3$	$R_1 = -\frac{3}{70}$ $R_2 = -\frac{3}{70}$	0	0.14 for R_1 0.14 for R_2	0.44 for R_1 0.42 for R_2

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