A NEW NON-SYMMETRIC INFORMATION DIVERGENCE OF CSISZAR'S CLASS, PROPERTIES AND ITS BOUNDS

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Abstract

Non-parametric measures give the amount of information supplied by the data for discriminating in favor of a probability distribution P against another Q, or for measuring the distance or affinity between P and Q.

There are several generalized functional divergences, such as: Csiszar divergence, Renyi- like divergence, Bregman divergence, Burbea- Rao divergence etc. all. In this paper, a non-parametric non symmetric measure of divergence which belongs to the family of Csiszár's f-divergence is proposed. Its properties are studied and get the bounds in terms of some well known divergence measures.

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1. INTRODUCTION

Let $\Gamma_n = \left\{ P = (p_1, p_2, p_3 \ldots, p_n) : p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, n \geq 2$ be the set of all complete finite discrete probability distributions. If we take $p_i \geq 0$ for some $i = 1, 2, 3, \ldots, n$, then we have to suppose that $0 f(0) = 0 f\left(\frac{0}{0}\right) = 0$.

Csiszar [2], given the generalized f- divergence measure, which is given by:

$$C_f(P,Q) = \sum_{i=1}^{n} q_i \, f\left(\frac{p_i}{q_i}\right) \tag{1.1}$$

Where $f : (0,\infty) \to R$ (set of real no.) is real, continuous and convex function and $P = (p_1, p_2, p_3 \ldots, p_n), Q = (q_1, q_2, q_3 \ldots, q_n) \in \Gamma_n$, where p_i and q_i are probability mass functions. Many known divergences can be obtained from this generalized measure by suitably defining the convex function f. Some of those are as follows:

❖ $K(P,Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right)$ = **Kullback- Leibler divergence measure [4]** (1.2)

❖ $\chi^2(P,Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}$ = **Chi- Square divergence measure [5]** (1.3)

❖ $h(P,Q) = \sum_{i=1}^{n} \frac{\left(\sqrt{p_i} - \sqrt{q_i}\right)^2}{2}$ = **Hellinger discrimination [3]** (1.4)

❖

❖ $R_a(P,Q) = \sum_{i=1}^{n} \frac{p_i^{\,a}}{q_i^{\,a-1}}, a > 1$ = **Renyi's "a" order entropy [6]** (1.5)

❖

❖ $B(P,Q) = \sum_{i=1}^{n} \sqrt{p_i \, q_i}$ = **Bhattacharya divergence measure [1]** (1.6)

❖

Relative information of type "s" [9]

❖ $\Phi_s(P,Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^{n} p_i^{\,s} q_i^{\,1-s} - 1 \right], s \neq 0, 1 \; and \; s \in R$ (1.7)

Particularly

$$\lim_{s\to 1}\Phi_s(P,Q)=K(P,Q),\ \lim_{s\to 0}\Phi_s(P,Q)=K(Q,P) \qquad (1.8)$$

Where $K(P,Q)$ is given by (1.2).

❖ $$G(P,Q)=\sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\log\left(\frac{p_i+q_i}{2\,p_i}\right) =$$

Relative AG Divergence [7] (1.9)

Similarly, we get many others divergences as well by defining suitable convex function

2. NEW INFORMATION DIVERGENCE MEASURE

In this section, we shall obtain a new divergence measure corresponding to new convex function, and will study the properties.

The following theorem is well known in literature **[2].**

Theorem 1: If the function f is convex and normalized, i.e., $f(1)=0$, then $C_f(P,Q)$ and its ad joint $C_f(Q,P)$ are both non-negative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.

Let f: (0, ∞) → R, be a mapping, defined as:

$$f(t)=\frac{(t-1)^4}{t},\ t\in(0,\infty) \qquad (2.1)$$

And

$$f'(t)=\frac{(t-1)^3}{t^2}(3t+1),\ f''(t)=\frac{2(t-1)^2}{t^3}\left(3t^2+2t+1\right) \qquad (2.2)$$

Properties of function defined by (2.1), are as follows:

- ❖ Since $f''(t)\geq 0\ \forall t\in(0,\infty)\Rightarrow f(t)$ is a convex function.
- ❖ Since $f(1)=0\Rightarrow f(t)$ is a normalized function.
- ❖ Since $f'(t)<0$ at $(0,1)$ and $f'(t)>0$ at $(1,\infty)\Rightarrow f(t)$ is monotonically decreasing in $(0,1)$ and monotonically increasing in $(1,\infty)$, and $f'(1)=0$.

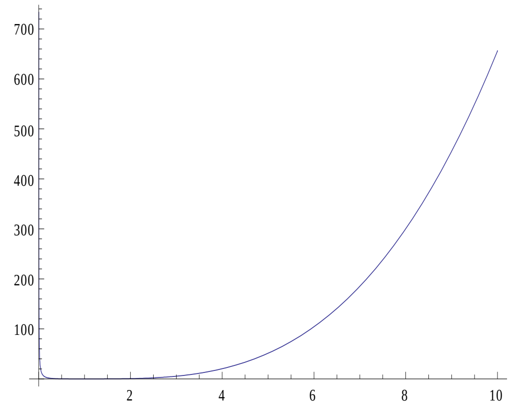

Fig1: Graph of convex function $f(t)$

Now, put (2.1) in (1.1), we get the following new divergence:

$$C_f(P,Q)=V^*(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^4}{p_i q_i^{\ 2}} \qquad (2.3)$$

Properties of divergence defined by (2.3), are as follows:

- ❖ In view of theorem 1, we can say that $V^*(P,Q)>0$ and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.
- ❖ $V^*(P,Q)=0\ if\ P=Q\ or\ p_i=q_i$ (Attains its minimum value).
- ❖ Since $V^*(P,Q)\neq V^*(Q,P)\Rightarrow V^*(P,Q)$ is non-symmetric divergence measure w.r.t. $P\ \&\ Q$.

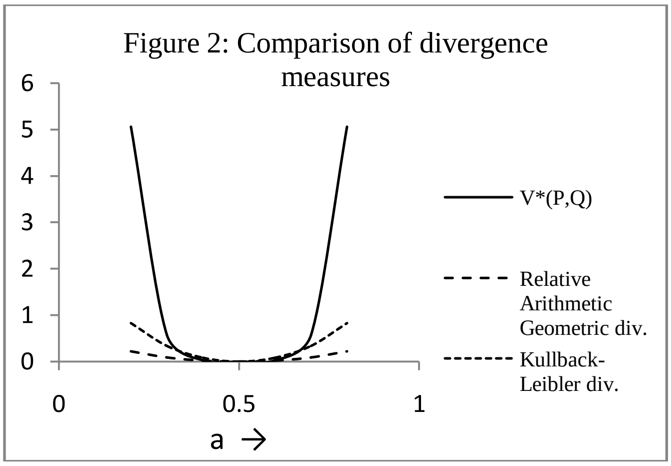

Figure 2: Comparison of divergence measures

Figure 2 shows the behavior of $V*(P,Q)$, Relative Arithmetic-Geometric divergence $G(P,Q)$ and Kullback-Leibler divergence $K(P,Q)$. We have considered $p_i=(a,1-a)\ and\ q_i=(1-a,a)\ where\ a\in(0,1)$. It is clear from figure 2 that the new measure $V^{*}(P,Q)$ has a steeper slope then $G(P,Q)$ *and* $K(P,Q)$.

3. CSISZAR'S FUNCTIONAL DIVERGENCE AND INEQUALITIES

The following theorem is well known in literature [8].

Theorem 2: Let $f: I \subset R_{+} \rightarrow R$ (I is an open interval) be a mapping which is normalized, i.e., $f(1)=0$ and suppose that

I. f is twice differentiable on $(\alpha,\beta), 0<\alpha\le 1\le\beta<\infty$ *with* $\alpha\neq\beta$.

II. There exist real constants m, M such that $m<M$ *and* $m\le t^{2-s} f''(t)\le M\ \forall t\in(\alpha,\beta)$ *and* $s\in R$ and

If $P,Q\in\Gamma_n$ *with* $0<\alpha\le \frac{p_i}{q_i}\le\beta<\infty\ \forall i=1,2,3...,n$

, then

$$m\,\Phi_s(P,\ Q)\le C_f(P,Q)\le\ M\,\Phi_s(P,\ Q) \tag{3.1}$$

And

$$m\left[\eta_s(P,Q)-\Phi_s(P,Q)\right]\le C_\rho(P,Q)-C_f(P,Q)\le M\left[\eta_s(P,Q)-\Phi_s(P,Q)\right] \tag{3.2}$$

Where

$$C_\rho(P,Q)=C_{f'}\left(\frac{P^2}{Q},P\right)-C_{f'}(P,Q)=\sum_{i=1}^{n}(p_i-q_i)\,f'\left(\frac{p_i}{q_i}\right) \tag{3.3}$$

$$\eta_s(P,Q)=C_{\Phi_s'}\left(\frac{P^2}{Q},P\right)-C_{\Phi_s'}(P,Q)=(s-1)^{-1}\sum_{i=1}^{n}(p_i-q_i)\left(\frac{p_i}{q_i}\right)^{s-1}, s\neq 1 \tag{3.4}$$

And $C_f(P,Q),\Phi_s(P,Q)$ are given by (1.1) and (1.7) respectively

4. BOUNDS OF NEW INFORMATION DIVERGENCE MEASURE

In this section, we derive bounds for $V^{*}(P,Q)$ in terms of the well known divergences in the following propositions at $s=2,1,1/2,0$ *and* -1, by using the theorem 2.

4.1 Proposition 4.1(at s=2)

Let $\chi^2(P,Q)$ *and* $V^{*}(P,Q)$ be defined as in (1.3) and (2.3) respectively. Then, we have

i. If $0<\alpha<1$, then

$$0\le V^{*}(P,Q)\le \max.\left\{\frac{(\alpha-1)^2}{\alpha^3}(3\alpha^2+2\alpha+1),\frac{(\beta-1)^2}{\beta^3}(3\beta^2+2\beta+1)\right\}\chi^2(P,Q)$$

$$0\le V_\rho^{*}(P,Q)-V^{*}(P,Q) \tag{4.1}$$

$$\le \max.\left\{\frac{(\alpha-1)^2}{\alpha^3}(3\alpha^2+2\alpha+1),\frac{(\beta-1)^2}{\beta^3}(3\beta^2+2\beta+1)\right\}\chi^2(P,Q) \tag{4.2}$$

ii. If $\alpha=1$, then

$$0\le V^{*}(P,Q)\le\frac{(\beta-1)^2}{\beta^3}(3\beta^2+2\beta+1)\chi^2(P,Q) \tag{4.3}$$

$$0\le V_\rho^{*}(P,Q)-V^{*}(P,Q)\le\frac{(\beta-1)^2}{\beta^3}(3\beta^2+2\beta+1)\chi^2(P,Q) \tag{4.4}$$

Proof:

Firstly, put s=2 in (1.7) and (3.4) respectively, we get

$$\Phi_s(P,Q)=\frac{1}{2}\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1=\frac{1}{2}\sum_{i=1}^{n}\frac{p_i^2}{q_i}-2p_i+q_i=\frac{1}{2}\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}=\frac{1}{2}\chi^2(P,Q) \tag{4.5}$$

$$\eta_s(P,Q)=\sum_{i=1}^{n}(p_i-q_i)\frac{p_i}{q_i}=\sum_{i=1}^{n}\frac{p_i^2}{q_i}-p_i=\sum_{i=1}^{n}\frac{p_i^2}{q_i}-1$$

$$=\sum_{i=1}^{n}\frac{p_i^2}{q_i}-2p_i+q_i=\sum_{i=1}^{n}\frac{(p_i-q_i)^2}{q_i}=\chi^2(P,Q) \tag{4.6}$$

And by putting $f'(t)$ in (3.3), we get

$$V_\rho^{*}(P,Q)=V_{f'}^{*}\left(\frac{P^2}{Q},P\right)-V_{f'}^{*}(P,Q)=\sum_{i=1}^{n}\frac{(p_i-q_i)^4}{(p_i\,q_i)^2}(3p_i+q_i) \tag{4.7}$$

Let $g(t)=f''(t)=\frac{2(t-1)^2}{t^3}(3t^2+2t+1)$ (After putting s=2 in $t^{2-s}f''(t)$)

Then $g'(t)=\frac{6\left(t^4-1\right)}{t^4}, g''(t)=\frac{24}{t^5}$

If $g'(t)=0 \Rightarrow t^4-1=0 \Rightarrow t=1,-1$

It is clear that g (t) is monotonic decreasing on (0, 1) and monotonic increasing on [1, ∞).

Also g (t) has minimum value at t=1, since $g''(1)=24>0$ so

$$m=\inf_{t\in(0,\infty)} g(t)=g(1)=0 \qquad (4.8)$$

Now, we have two cases:

i. If $0<\alpha<1$, then

$$M=\sup_{t\in(\alpha,\beta)} g(t)=\max\{g(\alpha),g(\beta)\}$$

$$=\max\left\{\frac{2(\alpha-1)^2}{\alpha^3}\left(3\alpha^2+2\alpha+1\right),\frac{2(\beta-1)^2}{\beta^3}\left(3\beta^2+2\beta+1\right)\right\} \qquad (4.9)$$

ii. If $\alpha=1$, then

$$M=\sup_{t\in[1,\beta)} g(t)=\frac{2(\beta-1)^2}{\beta^3}\left(3\beta^2+2\beta+1\right) \qquad (4.10)$$

The results (4.1), (4.2), (4.3) and (4.4) are obtained by using (2.3), (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) in 3.1 and 3.2.

4.2 Proposition 4.2(at s=1)

Let $K(P,Q)$ *and* $V^*(P,Q)$ be defined as in (1.2) and (2.3) respectively. Then, we have

i. If $0<\alpha<1$, then

$$0\le V^*(P,Q)\le \max.\left\{\frac{2(\alpha-1)^2}{\alpha^2}\left(3\alpha^2+2\alpha+1\right),\frac{2(\beta-1)^2}{\beta^2}\left(3\beta^2+2\beta+1\right)\right\}K(P,Q) \qquad (4.11)$$

$$0\le V_\rho^*(P,Q)-V^*(P,Q)$$
$$\le \max.\left\{\frac{2(\alpha-1)^2}{\alpha^2}\left(3\alpha^2+2\alpha+1\right),\frac{2(\beta-1)^2}{\beta^2}\left(3\beta^2+2\beta+1\right)\right\}K(Q,P) \qquad (4.12)$$

ii. If $\alpha=1$, then

$$0\le V^*(P,Q)\le\frac{2(\beta-1)^2}{\beta^2}\left(3\beta^2+2\beta+1\right)K(P,Q) \qquad (4.13)$$

$$0\le V_\rho^*(P,Q)-V^*(P,Q)\le\frac{2(\beta-1)^2}{\beta^2}\left(3\beta^2+2\beta+1\right)K(Q,P) \qquad (4.14)$$

Proof:
Firstly, put s=1 in (1.7) and (3.4) respectively, we get

$$\lim_{s\to1}\Phi_s(P,Q)=\sum_{i=1}^n p_i\log\left(\frac{p_i}{q_i}\right)=K(P,Q) \qquad (4.15)$$

$$\lim_{s\to1}\eta_s(P,Q)=\sum_{i=1}^n p_i\log\left(\frac{p_i}{q_i}\right)+q_i\log\left(\frac{q_i}{p_i}\right)=K(P,Q)+K(Q,P) \qquad (4.16)$$

Let $g(t)=t\ f''(t)=\frac{2(t-1)^2}{t^2}\left(3t^2+2t+1\right)$ (After putting s=1 in $t^{2-s}f''(t)$)

Then

$$g'(t)=\frac{4(t-1)}{t^3}\left(3t^3+t^2+t+1\right), g''(t)=\frac{12\left(t^4+1\right)}{t^4}$$

If

$$g'(t)=0\Rightarrow(t-1)\left(3t^3+t^2+t+1\right)=0\Rightarrow t=1,-0.63$$

It is clear that g (t) is monotonic decreasing on (0, 1) and monotonic increasing on [1, ∞).

Also g (t) has minimum value at t=1, since $g''(1)=24>0$ so

$$m=\inf_{t\in(0,\infty)} g(t)=g(1)=0 \qquad (4.17)$$

Now, we have two cases:

i. If $0<\alpha<1$, then

$$M=\sup_{t\in(\alpha,\beta)} g(t)=\max\{g(\alpha),g(\beta)\}$$

$$=\max\left\{\frac{2(\alpha-1)^2}{\alpha^2}\left(3\alpha^2+2\alpha+1\right),\frac{2(\beta-1)^2}{\beta^2}\left(3\beta^2+2\beta+1\right)\right\} \qquad (4.18)$$

ii. If $\alpha=1$, then

$$M=\sup_{t\in[1,\beta)} g(t)=\frac{2(\beta-1)^2}{\beta^2}\left(3\beta^2+2\beta+1\right) \qquad (4.19)$$

The results (4.11), (4.12), (4.13) and (4.14) are obtained by using (2.3), (4.7), (4.15), (4.16), (4.17), (4.18), and (4.19) in 3.1 and 3.2.

4.3 Proposition 4.3(at s=1/2)

Let $h(P,Q), R_a(P,Q), B(P,Q)$ and $V^*(P,Q)$ be defined as in (1.4), (1.5), (1.6) and (2.3) respectively. Then, we have

i. If $0<\alpha<1$, then

$$0 \le V^*(P,Q) \le \max\left\{\frac{8(\alpha-1)^2}{\alpha^{3/2}}\left(3\alpha^2+2\alpha+1\right), \frac{8(\beta-1)^2}{\beta^{3/2}}\left(3\beta^2+2\beta+1\right)\right\} h(P,Q) \tag{4.20}$$

$$0 \le V_\rho^*(P,Q) - V^*(P,Q)$$
$$\le \max\left\{\frac{8(\alpha-1)^2}{\alpha^{3/2}}\left(3\alpha^2+2\alpha+1\right), \frac{8(\beta-1)^2}{\beta^{3/2}}\left(3\beta^2+2\beta+1\right)\right\}\left[\frac{1}{2}\left\{R_{3/2}(Q,P)-B(P,Q)\right\}-h(P,Q)\right] \tag{4.21}$$

ii. If $\alpha=1$, then

$$0 \le V^*(P,Q) \le \frac{8(\beta-1)^2}{\beta^{3/2}}\left(3\beta^2+2\beta+1\right) h(P,Q) \tag{4.22}$$

$$0 \le V_\rho^*(P,Q) - V^*(P,Q)$$
$$\le \frac{8(\beta-1)^2}{\beta^{3/2}}\left(3\beta^2+2\beta+1\right)\left[\frac{1}{2}\left\{R_{3/2}(Q,P)-B(P,Q)\right\}-h(P,Q)\right] \tag{4.23}$$

Proof:
Firstly, put s=1/2 in (1.7) and (3.4) respectively, we get

$$\Phi_s(P,Q) = 4\sum_{i=1}^{n} 1-\sqrt{p_i q_i} = 2\sum_{i=1}^{n} 2-2\sqrt{p_i q_i}$$
$$= 2\sum_{i=1}^{n} p_i+q_i-2\sqrt{p_i q_i} = 4\sum_{i=1}^{n}\frac{\left(\sqrt{p_i}-\sqrt{q_i}\right)^2}{2} = 4h(P,Q) \tag{4.24}$$

$$\eta_s(P,Q) = 2\sum_{i=1}^{n}(q_i-p_i)\sqrt{\frac{q_i}{p_i}} = 2\sum_{i=1}^{n}\left(\frac{q_i^{3/2}}{p_i^{1/2}}-\sqrt{p_i q_i}\right) = 2\left[R_{3/2}(Q,P)-B(P,Q)\right] \tag{4.25}$$

Let $g(t) = t^{\frac{3}{2}} f''(t) = \frac{2(t-1)^2}{t^{\frac{3}{2}}}\left(3t^2+2t+1\right)$ (After putting s=1/2 in $t^{2-s} f''(t)$)

Then $g'(t) = \frac{3(t-1)}{t^{\frac{5}{2}}}\left(5t^3+t^2+t+1\right), g''(t) = \frac{3}{2t^{\frac{7}{2}}}\left(15t^4-4t^3+5\right)$

If $g'(t)=0 \Rightarrow (t-1)\left(5t^3+t^2+t+1\right)=0 \Rightarrow t=1,-0.53$

It is clear that g (t) is monotonic decreasing on (0, 1) and monotonic increasing on [1, ∞).

Also g (t) has minimum value at t=1, since $g''(1)=24>0$ so

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = 0 \tag{4.26}$$

Now, we have two cases:

i. If $0<\alpha<1$, then

$$M = \sup_{t\in(\alpha,\beta)} g(t) = \max\left\{g(\alpha), g(\beta)\right\}$$
$$= \max\left\{\frac{2(\alpha-1)^2}{\alpha^{\frac{3}{2}}}\left(3\alpha^2+2\alpha+1\right), \frac{2(\beta-1)^2}{\beta^{\frac{3}{2}}}\left(3\beta^2+2\beta+1\right)\right\} \tag{4.27}$$

ii. If $\alpha=1$, then

$$M = \sup_{t\in[1,\beta)} g(t) = \frac{2(\beta-1)^2}{\beta^{\frac{3}{2}}}\left(3\beta^2+2\beta+1\right) \tag{4.28}$$

The results (4.20), (4.21), (4.22) and (4.23) are obtained by using (2.3), (4.7), (4.24), (4.25), (4.26), (4.27), and (4.28) in 3.1 and 3.2.

4.4 Proposition 4.4(at s=0)

Let $K(P,Q), \chi^2(P,Q)$ and $V^*(P,Q)$ be defined as in (1.2), (1.3) and (2.3) respectively. Then, we have

i. If $0<\alpha<1$, then

$$0 \le V^*(P,Q) \le \max\left\{\frac{2(\alpha-1)^2}{\alpha}\left(3\alpha^2+2\alpha+1\right), \frac{2(\beta-1)^2}{\beta}\left(3\beta^2+2\beta+1\right)\right\} K(Q,P) \tag{4.29}$$

$$0 \le V_\rho^*(P,Q) - V^*(P,Q)$$
$$\le \max\left\{\frac{2(\alpha-1)^2}{\alpha}\left(3\alpha^2+2\alpha+1\right), \frac{2(\beta-1)^2}{\beta}\left(3\beta^2+2\beta+1\right)\right\}\left\{\chi^2(Q,P)-K(Q,P)\right\}$$
(4.30)

ii. If $\alpha = 1$, then

$$0 \le V^*(P,Q) \le \frac{2(\beta-1)^2}{\beta}\left(3\beta^2+2\beta+1\right)K(Q,P)$$
(4.31)

$$0 \le V_\rho^*(P,Q) - V^*(P,Q) \le \frac{2(\beta-1)^2}{\beta}\left(3\beta^2+2\beta+1\right)\left\{\chi^2(Q,P)-K(Q,P)\right\}$$
(4.32)

Proof:
Firstly, put s=0 in (1.7) and (3.4) respectively, we get

$$\lim_{s\to 0}\Phi_s(P,Q) = \sum_{i=1}^n q_i \log\left(\frac{q_i}{p_i}\right) = K(Q,P)$$
(4.33)

$$\eta_s(P,Q) = \sum_{i=1}^n \frac{q_i^2}{p_i} - q_i = \sum_{i=1}^n \frac{q_i^2}{p_i} - 1 = \sum_{i=1}^n \frac{q_i^2}{p_i} - 2q_i + p_i = \sum_{i=1}^n \frac{(p_i-q_i)^2}{p_i} = \chi^2(Q,P)$$
(4.34)

Let $g(t) = t^2 f''(t) = \frac{2(t-1)^2}{t}\left(3t^2+2t+1\right)$ (After putting s=0 in $t^{2-s} f''(t)$)

Then

$$g'(t) = \frac{2(t-1)}{t^2}\left(9t^3+t^2+t+1\right), g''(t) = \frac{1}{t^3}\left(36t^4-16t^3+4\right)$$

If

$$g'(t) = 0 \Rightarrow (t-1)\left(9t^3+t^2+t+1\right) = 0 \Rightarrow t = 1, -0.43$$

It is clear that g (t) is monotonic decreasing on (0, 1) and monotonic increasing on [1, ∞).

Also g (t) has minimum value at t=1, since $g''(1) = 24 > 0$ so

$$m = \inf_{t\in(0,\infty)} g(t) = g(1) = 0 \quad (4.35)$$

Now, we have two cases:

i. If $0 < \alpha < 1$, then

$$M = \sup_{t\in(\alpha,\beta)} g(t) = \max\{g(\alpha), g(\beta)\}$$
$$= \max\left\{\frac{2(\alpha-1)^2}{\alpha}\left(3\alpha^2+2\alpha+1\right), \frac{2(\beta-1)^2}{\beta}\left(3\beta^2+2\beta+1\right)\right\}$$
(4.36)

ii. If $\alpha = 1$, then

$$M = \sup_{t\in[1,\beta)} g(t) = \frac{2(\beta-1)^2}{\beta}\left(3\beta^2+2\beta+1\right)$$
(4.37)

The results (4.29), (4.30), (4.31) and (4.32) are obtained by using (2.3), (4.7), (4.33), (4.34), (4.35), (4.36), and (4.37) in 3.1 and 3.2.

4.5 Proposition 4.5(at s =-1)

Let $\chi^2(P,Q), \mathrm{R}_a(P,Q)$ *and* $V^*(P,Q)$ be defined as in (1.3), (1.5) and (2.3) respectively. Then, we have

i. If $0 < \alpha < 1$, then

$$0 \le V^*(P,Q) \le \max\left\{(\alpha-1)^2\left(3\alpha^2+2\alpha+1\right), (\beta-1)^2\left(3\beta^2+2\beta+1\right)\right\}\chi^2(Q,P)$$

$$0 \le V_\rho^*(P,Q) - V^*(P,Q) \quad (4.38)$$

$$\le \max\left\{(\alpha-1)^2\left(3\alpha^2+2\alpha+1\right), (\beta-1)^2\left(3\beta^2+2\beta+1\right)\right\}\left\{R_2(Q,P)-R_3(Q,P)-\chi^2(Q,P)\right\}$$
(4.39)

ii. If $\alpha = 1$, then

$$0 \le V^*(P,Q) \le (\beta-1)^2\left(3\beta^2+2\beta+1\right)\chi^2(Q,P)$$
(4.40)

$$0 \le V_\rho^*(P,Q) - V^*(P,Q) \le (\beta-1)^2\left(3\beta^2+2\beta+1\right)\left\{R_3(Q,P)-R_2(Q,P)-\chi^2(Q,P)\right\}$$
(4.41)

Proof:
Firstly, put s=-1 in (1.7) and (3.4) respectively, we get

$$\Phi_s(P,Q) = \frac{1}{2}\sum_{i=1}^n \frac{q_i^2}{p_i} - 1 = \frac{1}{2}\sum_{i=1}^n \frac{q_i^2}{p_i} - 2q_i + p_i = \frac{1}{2}\sum_{i=1}^n \frac{(p_i-q_i)^2}{p_i} = \frac{1}{2}\chi^2(Q,P)$$
(4.42)

$$\eta_s(P,Q)=\frac{1}{2}\sum_{i=1}^{n}(q_i-p_i)\frac{q_i^2}{p_i^2}=\frac{1}{2}\sum_{i=1}^{n}\left(\frac{q_i^3}{p_i^2}-\frac{q_i^2}{p_i}\right)=\frac{1}{2}\left[R_3(Q,P)-R_2(Q,P)\right] \tag{4.43}$$

Let $g(t)=t^3 f''(t)=2(t-1)^2\left(3t^2+2t+1\right)$ (After putting s=-1 in $t^{2-s} f''(t)$)

Then $g'(t)=24\,t^2\,(t-1), g''(t)=72\,t^2-48t$

If $g'(t)=0 \Rightarrow t=0,1$

It is clear that g (t) is monotonic decreasing on (0, 1) and monotonic increasing on [1, ∞).

Also g (t) has minimum value at t=1, since $g''(1)=24>0$ so

$$m=\inf_{t\in(0,\infty)} g(t)=g(1)=0 \tag{4.44}$$

Now, we have two cases:

i. If $0<\alpha<1$, then

$$M=\sup_{t\in(\alpha,\beta)} g(t)=\max\left\{g(\alpha),g(\beta)\right\}$$
$$=\max\left\{2(\alpha-1)^2\left(3\alpha^2+2\alpha+1\right),2(\beta-1)^2\left(3\beta^2+2\beta+1\right)\right\} \tag{4.45}$$

ii. If $\alpha=1$, then

$$M=\sup_{t\in[1,\beta)} g(t)=2(\beta-1)^2\left(3\beta^2+2\beta+1\right) \tag{4.46}$$

The results (4.38), (4.39), (4.40) and (4.41) are obtained by using (2.3), (4.7), (4.42), (4.43), (4.44), (4.45), and (4.46) in 3.1 and 3.2.

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