# METHOD OF SOLVING MULTI-OBJECTIVE OPTIMIZATION PROBLEM IN THE PRESENCE OF UNCERTAINTY

**Mingliqulov Zafar**

*Centre for development of software and hardware - software systems at the Tashkent University of Information Technologies, Uzbekistan*

## Abstract

*Multi-objective formulations are realistic models for many complex engineering optimization problems. In many real-life problems considered objectives conflict with each other and optimization of one target solutions can lead to unacceptable results for other purposes. A reasonable solution of the multi-objective problem is to study the set of solutions, each of which satisfies the objectives at an acceptable level with no dominance of any of the solutions. The article provides a brief overview of multi-objective optimization methods (by Pareto criteria) and their improvement.*

***Keywords:*** *A fuzzy set, forecasting, risk, weakly formalized process for Pareto multi-objective optimization.*

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## 1. INTRODUCTION

The goal of this work is to present an overview of the methods of multi-criteria optimization on Pareto criteria and its improvement. In most cases objectives of the multi-objective problem conflict with each other, thereby preventing simultaneous optimization of each objective. Many, if not all, real engineering problems actually have more than one objective, that is, minimize costs, maximize productivity, increase reliability, etc. They are challenging, but real problems.

There are two approaches to the multi-criteria optimization. One of them is the convolution of the individual objective functions into a single component or to impose on all criteria except one the set of restrictions. In the first case the determination of the total objective function is possible with the help of the theory of utility, the weighted sum, etc., but the problem lies in the proper choice of weights or utility functions that characterize the preferences of the customer. This optimization method doesn't provide a set, but a unique solution, which can be examined to identify possible trade-offs.

The second approach is to define a set of Pareto optimal solutions, or a representative subset. Pareto solution is the set of non-dominant to each other solutions. When moving from one Pareto solutions to another, there are always some loss in one solution, and certain improvements in others. Pareto-optimal set of solutions are often preferred by single results because they are more practical when considering the trade-offs in the real life problems. The set of Pareto solutions may be any size, but typically the size increases with increasing number of purposes.

Note that the Pareto principle is not universal and applies only if a number of axioms are met. And even if these axioms are met, the construction of the Pareto set can cause significant difficulties [1, 7-8].

## 2. STATEMENT OF THE PROBLEM OF MULTI-OBJECTIVE OPTIMIZATION

In the formulation of multi-objective optimization problem mandatory condition to meet all individual criteria and constraints is introduced as a requirement for the optimality of solutions, namely all of the functions belonging to the set of optimal solutions must be different from zero in the optimum point, the optimum criteria must be satisfied to the extent possible. The increase in the value of the generalized criterion should not happen while improving the value of some indicators of quality at the expense of the deterioration of others. In the terminology of decision theory, the latter requirement is equivalent to the condition of belonging to a Pareto optimum [1].

The problem of multi-criteria optimization has the following form:

$$f(x)=\left[f_1(x), f_2(x),..., f_q(x)\right]^T \to \min,$$
$$x \in X, \qquad (1)$$

Where

$$f_k(x) = \sum_{j=1}^{n} c_{kj} x_j,$$
$$k \in Q = \{1,2,..., q\},$$
$$X = \{x \in R^n \mid Ax \geq b, x \geq 0\},$$
$$\left( \sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = \overline{1,m} \right).$$

The problem of multi-objective optimization with fuzzy goal involves finding such x that satisfy the following constraints [2-4]:

$$f_k(x) \leq \tilde{g}_k, \; k = 1,2,..., q, \quad x \in X, \tag{2}$$

Where $\tilde{g}_k$ - fuzzy set,

$$\mu_k(f_k(x)) = \begin{cases} 1, & f_k(x) \leq g_k, \\ 1 - \dfrac{f_k(x) - g_k}{t_k}, & g_k \leq f_k(x) \leq g_k + t_k, \\ 0, & f_k(x) \geq g_k + t_k. \end{cases} \tag{3}$$

The solving the Fuzzy problem (2) can be reduced to the solving the following unfuzzy problem:

$$\lambda \to \max,$$
$$\mu_k(f_k(x)) \geq \lambda,$$
$$x \in X. \tag{4}$$

## 3. FINDING THE PARETO OPTIMAL SOLUTIONS

The solution $x^0 \in X$ is called Pareto optimal solution if $\mu_k(f_k(y)) \leq \mu_k(f_k(x^0))$ for all y, and $\mu_S(f_S(y)) < \mu_S(f_S(x^0))$ for at least one.

The solution $x^0 \in X$ is called optimal solution on the Pareto type criteria if there is no $y \in X$ better than $x^0$ on the Pareto type criteria.

We introduce the concept the improvability of solution $y \in X$ on the Pareto type criteria in fuzzy environment: we call solution $y \in X$ improvable if there is the solution $x^0 \in X$, which is better than y on the Pareto type criteria.

The solution $x^0 \in X$ is improvable in finding multi-objective fuzzy solutions $f(x) = [f_1(x), f_2(x),..., f_p(x)]$ if and only if there is a vector $\gamma \in R^Q$, for which the following inequalities are met: $\mu_k(f_k(x^0)) \leq c^k$, $\mu_S(f_S(x^0)) < c^S$

For all $k \in \{1,..., Q\}$ and at least one $S \in \{1,..., Q\}$, where $c^k = c - \gamma_k$, $c = \max_y \min_k [\mu_k(f_k(y)) + \gamma_k]$.

Let required inequalities are satisfied, then according to the definition of $c^k$ $y \in X$, for which the inequality $c \leq \mu_k(f_k(y)) + \gamma_k$, and therefore $c^k \leq \mu_k(f_k(y))$, $\mu_S(f_S(x^0)) < c^S \leq \mu_S(f_S(y))$, $\mu_S(f_S(x^0)) \leq c^k \leq \mu_k(f_k(y))$ are met for all $k \in \{1,..., Q\}$ or at least one $S \in \{1,..., Q\}$, exists. These inequalities show that the solution $x^0 \in X$ is improvable.

Let solution $x^0 \in X$ is improvable and let $y \in X$ is the solution, which is better than the solution $x^0$ on Pareto criteria. We assume that $\gamma_k = \mu_S(f_S(y)) - \mu_k(f_k(y))$ for all $k \in \{1,..., Q\}$, where $S : \mu_S(f_S(y)) > \mu_S(f_S(x^0))$. Then

$$\max_k [\mu_k(f_k(y)) + \gamma_k] = \min_y [\mu_k(f_k(y)) + \gamma_k] = \mu_S(f_S(y)).$$

Considering $\min_k [\mu_k(f_k(y)) + \gamma_k] \leq c$ for all $\gamma$ form $R^Q$, we get $\mu_k(f_k(x^0)) + \gamma_k \leq \mu_k(f_k(y)) + \gamma_k \leq \max_k [\mu_k(f_k(y)) + \gamma_k] =$ $= \min_y [\mu_k(f_k(y)) + \gamma_k] \leq c$ For all $k \in \{1,..., Q\}$ and $\mu_S(f_S(x^0)) + \gamma_S < \mu_S(f_S(y)) + \gamma_S \leq c$ for at least one $S \in \{1,..., Q\}$. Thus the validity of these inequalities is proved.

Solution $x^0 \in X$ is improvable in situation of making multipurpose decisions if and only if there exists a vector γ from the set $\Gamma = \{\gamma \in R^Q : \max_y \mu_k(f_k(y)) - \min_y \mu_\rho(f_\rho(y)) \geq \gamma_\rho - \gamma_k,$ $(\rho, k = 1,..., Q, \rho \neq k)\}$ such that inequalities $\mu_k(f_k(x^0)) \leq c^k$, $\mu_S(f_S(x^0)) < c^S$ are met. The validity of these inequalities follows from the following considerations $[\mu_k(f_k(x^0)) + \gamma_k] \leq (\mu_k(f_k(y)) + \gamma_k) \leq \max_k [\mu_k(f_k(y)) + \gamma_k] = \min_\rho [\mu_\rho(f_\rho(y)) + \gamma_\rho] \leq c,$ for every $k, \rho \in \{1,..., Q\}$, $(\mu_S(f_S(x^0)) + \gamma_S) < (\mu_S(f_S(y)) + \gamma_S) \leq c$.

It follows that if the evaluation functionals $\{\mu_k(f_k(y))\}_{k=1}^{Q}$ are obtained after the application of the natural normalization, then the domain Γ has following form:

$$\Gamma = \{\gamma \in R^Q; |\gamma_\rho - \gamma_k| \leq 1, \quad k, \rho = 1,..., Q; \rho \neq k\}.$$

Thus, the solution to the problem about improvability, Pareto optimality of multi-purpose solution $x^0 \in X$ on Pareto criterion is reduced to the existence (absence) of a vector $\gamma \in \Gamma$ for which the inequalities $\mu_k\left(f_k\left(x^0\right)\right) \le c^k$ $\mu_S\left(f_S\left(x^0\right)\right) < c^S$ are met.

Thus the following inequalities should be met (be incompatible) for the solution $y \in X$ to be improvable (optimal on Pareto):

$$\mu_k\left(f_k(y)\right) \le \max_{\gamma \in \Gamma}\left\{\max_z \min_\rho \left[\mu_\rho\left(f_\rho(z)\right)+\gamma_\rho\right]-\gamma_k\right\}, (k=1,\dots,Q).$$

The validity of these inequalities follows from the following considerations:

$$\mu_k\left(f_k\left(x^0\right)\right)+\gamma_k \le \mu_k\left(f_k(y)\right)+\gamma_k \le \max_z\left[\mu_k\left(f_k(z)\right)+\gamma_k\right]=\min_\rho\left[\mu_\rho\left(f_\rho(z)\right)+\gamma_\rho\right] \le$$

$$\le \max_z \min_\rho\left[\mu_\rho\left(f_\rho(z)\right)+\gamma_\rho\right] \le \max_{\gamma \in \Gamma}\left\{\max_z \min_\rho\left[\mu_\rho\left(f_\rho(z)\right)+\gamma_\rho\right]\right\},$$

$$\left(\mu_S\left(f_S\left(x^0\right)\right)+\gamma_S\right) < \left(\mu_S\left(f_S(y)\right)+\gamma_S\right) \le \max_{\gamma \in \Gamma}\left\{\min_z \min_\rho\left(\mu_S\left(f_S(z)\right)+\gamma_\rho\right)\right\}$$

for all $(\rho, k=1,\dots,Q, \rho \ne k)$ or at least one $S \in \{1,\dots,Q\}$.

Let $\mu_k\left(f_k(y)\right)$ be the membership function of $f_k(y)$, which is defined as (3). $x^0$ is optimal solution of improvable problem

$$\sum_{k=1}^{Q}\gamma_k \to \max,\ \mu_k\left(f_k(x)\right)-\gamma_k \ge \lambda^* \ \ k=1,\dots,Q,\ x \in X,\ \ \gamma_k \ge 0. \tag{5}$$

Then solution $x^0 \in X$ is Pareto optimal solution of the problem (1).

We suppose inverse case. Let $x^0 \in X$ be Pareto optimal solution of (1). Then there exists such solution $y \in X$ that $f_k(y) \le f_k\left(x^0\right)$ for all ($k=1,\dots,Q$) and $f_S(y) < f_S\left(x^0\right)$ for some $S \in \{1,\dots,Q\}$. Since the vector $\gamma_k, k=1,\dots,Q$ is positive and satisfies the following equations:

$$\mu_k\left(f_k\left(x^0\right)\right)-\gamma_k=\lambda^*,\ k=1,\dots,Q$$

And

$$\sum_{k=1}^{Q}\gamma_k=\sum_{k=1}^{Q}\mu_k\left(f_k\left(x^0\right)\right)-Q\lambda^*.$$

Nevertheless, there is $f_k(y) \le f_k\left(x^0\right)$ for all ($k=1,\dots,Q$) and $f_S(y) \le f_S\left(x^0\right)$ for some $S$.

This leads to the following inequality:

$$\sum_{k=1}^{Q}\gamma_k=\sum_{k=1}^{Q}\mu_k\left(f_k\left(x^0\right)\right)-Q\lambda^*=\sum_{\substack{k=1\\k\ne S}}^{Q}\mu_k\left(f_k\left(x^0\right)\right)+\mu_S\left(f_S\left(x^0\right)\right)-Q\lambda^*<$$

$$<\sum_{\substack{k=1\\k\ne S}}^{Q}\mu_k\left(f_k(y)\right)+\mu_S\left(f_S(y)\right)-Q\lambda^*.$$

This means that the solution (5) $x^0 \in X$ is not optimal.

Finding Pareto optimal solution (5) in condition of (3) leads to solving following problem of linear programming:

$$\begin{cases} R=\sum_{k=1}^{q}c_k x_k \to \max, \\ \sum_{j=1}^{n}a_{kj}x_j \le b_k,\ k=1,\dots,q+m, \\ x_j \ge 0, \quad j=1,\dots,n. \end{cases}$$

We solve this problem using recurrent neural networks.

For calculating the model with recurrent neural networks moving to the opposite function $R=-\sum_{k=1}^{q}c_k x_k$ and to write the value of $c_k$ in the appropriate table with opposite sign is needed.

When vector is given to the input of the network, the state of neurons are determined, but then because the outputs of neurons are the feedbacks, to their inputs again a new vector is entered and the state changes again. The concept of stability is connected with recurrent networks [6]. The network is considered to be stable if, after a finite number of iterations, the neurons receive state that no further changes. When a vector is given to the input of stable recurrent networks, the output signals of neurons are generated, which are then applied to the inputs again, generating a new state vector again, but while the network is not set to the final state, with the increasing number of iterations, the number of changes of nodes' states is reduced. Network without feedback is always stable, since when one vector is given to the input, nodes of the network change their states only once due to the constancy of input neurons.

For solving the problem (1)-(2) recurrent neural network, which is defined with the following differential equation, is proposed:

$$\frac{\partial u_k(t)}{\partial t} = -\eta\left(\sum_{j=1}^{n} a_{kj}x_j - b_k\right) + \lambda c_k \exp\left(-\frac{t}{\tau}\right),$$

Where $x_k = f(u_k(t)), \quad f(u) = \frac{1}{1+\exp(-\beta u)}$. As in the Hopfield network, it uses neurons matrix with the size of $n \times n$, but the neurons interact not on the principle "with each other", but by the rows and columns.

The difference version of this equation has the following form:

$$u_k^{t+1} = u_k^t - \Delta t \cdot \left[\eta\left(\sum_{j=1}^{n} a_{kj}x_j - b_k\right) - \lambda c_k \exp\left(-\frac{t}{\tau}\right)\right] \quad (6)$$

Where $\Delta t$ - step by time. Parameters $\Delta t, \eta, \lambda, \tau, \beta$ are selected experimentally and they significantly impact to the speed of problem solving and to the quality of solution.

To speed up the solving of equations (6) the principle of «Winner takes all» is proposed [2-4]:

Matrix $\|x_k^0\|$ of random values $x_k^0 \in [0,1]$ is generated.

Iteration (6) is continued until following inequality is met:

$$\sum_{j=1}^{n} a_{kj}x_j - b_k \le \varepsilon,$$

Where $\varepsilon$ - the specified accuracy of the constraints

Steps 1 and 2 are repeated.

One of the main properties of artificial neural networks is the reliability of the neural network models. This property may allow building fo neural network systems for areas where high reliability is required.

No less important is the learning property of neural networks. Because of this property, not only they can recognize patterns applied to their input, but also adjust their parameters with the help of the appropriate procedure to carry out correct recognition as much as possible. Thus, the neural networks can operate in two modes or phases: a learning mode and the recognition mode.

The ability of neural networks to generalize is also very important property. Due to this property not only the network can show the imaged given during the training, but also build new ones. This increases the "competence" of systems based on neural networks.

## 4. COMPUTATIONAL EXPERIMENT

We consider an example taken from [5]. For solving multi-criteria problem

$$\begin{cases} z_1(x) = 4x_1 - 6x_2 \to \min \\ z_2(x) = -2x_1 - x_2 \to \min \\ -x_1 + x_2 \le 3 \\ x_2 \le 5 \\ x_1 + x_2 \le 10 \\ x_1 \le 8 \end{cases} \quad (7)$$

With the following fuzzy objectives:

$$g_1 = 20; \quad g_2 = -9$$

And tolerances

$$t_1 = 2; \qquad t_2 = 2$$

We form following membership functions:

$$\mu_1(z_1(x)) = \begin{cases} \dfrac{2r - z_1(x)}{2} & z_1(x) \le 2r \\ 0 & z_1(x) \ge 2r \end{cases}$$

$$\mu_1(z_1(x)) = \begin{cases} \dfrac{-7 - z_2(x)}{2} & z_1(x) \le -7 \\ 0 & z_1(x) \ge -7 \end{cases}$$

Then model (7) takes the following form:

$$\begin{cases} \lambda \to \max \\ \frac{1}{2}(22 - (4x_1 - 6x_2)) \ge \lambda \\ \frac{1}{2}(-7 - (-2x_1 - x_2)) \ge \lambda \\ -x_1 + x_2 \le 3 \\ x_2 \le 5 \\ x_1 + x_2 \le 10 \\ x_1 \le 8 \end{cases}$$

The solution of this model is

$$x^* = \left(x_1^*, x_2^*\right) = (4{,}5; 0),$$

Optimal value is $\lambda^* = 1$, the values of the objective function are

$$z\left(x^*\right) = 18, \qquad z\left(x^*\right) = -9.$$

We consider the existence of the vector $\gamma = (\gamma_1, \gamma_2)$, which may improve the solution $x^*$.

For this we solve the following problem:

$$\begin{cases} \gamma_1 + \gamma_2 \to \max \\ 4x_1 - 6x_2 + 2\gamma_1 \le 20 \\ -2x_1 - x_2 + 2\gamma_2 \le -9 \\ -x_1 + x_2 \le 3 \\ x_2 \le 5 \\ x_1 + x_2 \le 10 \\ x_1 \le 8 \end{cases} \qquad (8)$$

The solution of the model (8) is

$$x^{**} = \left(x_1^{**},\ x_2^{**}\right) = (2;5)$$

Optimal values are:

$\gamma_1 = 21;\ \ \gamma_2 = 0$, the values of the objective function are

$$z_1(x^{**}) = -22; \quad z_2(x^{**}) = -9.$$

Thus there is a vector $\gamma = (\gamma_1, \gamma_2)$, where the Pareto-optimal solution is improvable, i.e.

$$z_1\left(x^{**}\right) < z_1\left(x^*\right) \text{ и } z_2\left(x^{**}\right) = z_2\left(x^*\right)$$

## 5. CONCLUSIONS

Thus, the solving the problem of Pareto optimality of the multi-objective solution $x^0 \in R^n$ on Pareto criteria or its improvability is reduced to establishing the existence or absence of a vector that satisfies the inequalities

$$\mu_k\left(f_k\left(x^0\right)\right) \le c^k \quad \mu_s\left(f_s\left(x^0\right)\right) < c^S.$$

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## BIOGRAPHIES

He got his BSc and MSc degrees from "Software engineering for Computer Systems, networks and complexes" department, Tashkent University of Information Technology in 2005 and 2007 respectively. He published several papers on the topic of Data mining, such as fuzzy sets theory and neural networks in many international conferences and journals. Currently, He is working as senior researcher at Centre for development of software and hardware - software systems at the Tashkent University of Information Technologies.