BOUNDS ON INVERSE DOMINATION IN SQUARES OF GRAPHS

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Abstract

Let D be a minimum dominating set of a square graph G^2. If $V(G^2) - D$ contains another dominating set D' of G^2, then D' is called an inverse dominating set with respect to D. The minimum cardinality of vertices in such a set is called an inverse domination number of G^2 and is denoted by $\gamma^{-1}(G^2)$. In this paper, many bounds on $\gamma^{-1}(G^2)$ were obtained in terms of elements of G. Also its relationship with other domination parameters was obtained.

Key words: *Square graph, dominating set, inverse dominating set, Inverse domination number.*

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1. INTRODUCTION

In this paper, we follow the notations of [1]. We consider only finite undirected graphs without loops or multiple edges. In general, we use $< X >$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v.

The minimum (maximum) degree among the vertices of G is denoted by $\delta(G)\big(\Delta(G)\big)$. A vertex of degree one is called an end vertex. The term $\alpha_0(G)(\alpha_1(G))$ denotes the minimum number of vertices(edges) cover of G. Further, $\beta_0(G)(\beta_1(G))$ represents the vertex(edge) independence number of G.

A vertex with degree one is called an end vertex. The distance between two vertices u and v is the length of the shortest uv-path in G. The maximum distance between any two vertices in G is called the diameter of G and is denoted by $diam(G)$.

The square of a graph G denoted by G^2, has the same vertices as in G and the two vertices u and v are joined in G^2 if and only if they are joined in G by a path of length one or two. The concept of squares of graphs was introduced in [2].

A set $S \subseteq V(G)$ is said to be a dominating set of G, if every vertex in $(V - S)$ is adjacent to some vertex in S. The minimum cardinality of vertices in such a set is called the domination number of G and is denoted by $\gamma(G)$. Further, if the subgraph $< S >$ is independent, then S is called an independent dominating set of G. The independent omination number of G, denoted by $\gamma_i(G)$ is the minimum cardinality of an independent dominating set of G.

A dominating set S of G is said to be a connected dominating set, if the subgraph $< S >$ is connected in G. The minimum cardinality of vertices in such a set is called the connected domination number of G and is denoted by $\gamma_c(G)$.

A dominating set S is called total dominating set, if for every vertex $v \in V$, there exists a vertex $u \in S,\ u \notin v$ such that u is adjacent to v. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of G.

Further, a dominating set S is called an end dominating set of G, if S contains all the end vertices in G. The minimum cardinality of vertices in such a set is called the end domination number of G and is denoted by $\gamma_e(G)$.

Domination related parameters are now well studied in graph theory (see [3] and [9]).

Let S be a minimum dominating set of G, if the compliment $V - S$ of S contains a dominating set S', then S' is called an inverse dominating set of G with respect to S. The minimum cardinality of vertices in such a set is called an inverse domination number of G and is denoted by $\gamma^{-1}(G)$. Inverse domination was introduced by V. R. Kulli and S. C. Sigarkanti [10].

A set $D \subseteq V(G^2)$ is said to be a dominating set of G^2, if every vertex not in D is adjacent to a vertex in D. The minimum cardinality of vertices in such a set is called the domination number of G^2 and is denoted by $\gamma(G^2)$(see [4]).

A set D of G^2 is said to be connected dominating set of G^2 if every vertex not in D is adjacent to at least one vertex in D and the set induced by D is connected. The minimum cardinality of a connected dominating set of G^2 is called connected domination number of G^2(see [5]).

A dominating set D of G^2 is said to be total dominating set, if for every vertex $v \in V(G^2)$, there exists a vertex $u \in D, u \neq v$, such that u is adjacent to v. The total domination number of G^2 denoted by $\gamma_t(G^2)$ is the minimum cardinality of a total dominating set of G^2(see [6]).

A dominating set D of G^2 is said to be restrained dominating set, if for every vertex not in D is adjacent to a vertex in D and to a vertex in $(V - D)$. The restrained domination number of G^2, denoted by $\gamma_{re}(G^2)$ is the minimum cardinality of a restrained dominating set of G^2. Further, a dominating set D of G^2 is said to be double dominating set, if for every vertex $v \in V(G^2)$ is dominated by at least two vertices of D. The double domination number of G^2, denoted by $\gamma_d(G^2)$ is the minimum cardinality of a double dominating set of G^2(see [7] and [8]).

Analogously, let D be a minimum dominating set of a square graph G^2. If $V(G^2) - D$ contains another dominating set D' of G^2, then D' is called an inverse dominating set with respect to D. The minimum cardinality of vertices in such a set is called an inverse domination number of G^2 and is denoted by $\gamma^{-1}(G^2)$. In this paper, many bounds on $\gamma^{-1}(G^2)$ were obtained in terms of elements of G. Also its relationship with other domination parameters was obtained.

2. RESULTS

Theorem 2.1: For any connected graph G, $\gamma^{-1}(G^2) + \gamma_t(G^2) \leq \delta(G) + \gamma_t(G)$.

Proof: Let $S = \{v_1, v_2, \dots, v_m\} \subseteq V(G)$ be the minimum set of vertices which covers all the vertices in G. Suppose the subgraph $< S >$ has no isolated vertices, then S itself is a γ_t -set of G. Otherwise, let $B = \{v_1, v_2, \dots, v_n\} \subseteq S$ be the set of vertices with $\deg(v_i) = D, 1 \leq i \leq n$. Now make $\deg(v_i) = 1$ by adding vertices $\{u_i\} \subseteq V(G) - S$ and $N(v_i) \in \{u_i\}$. Clearly, $S_1 = S \cup B \cup \{u_i\}$ forms a minimal total dominating set of G. Since $V(G) = V(G^2)$, let $D_t = \{v_1, v_2, \dots, v_k\} \subseteq S_1$ be the minimal γ_t-set of G^2. Suppose $D \subseteq S$ is a minimal dominating set of G^2. Then there exists a vertex set $D' = \{v_1, v_2, \dots, v_k\} \subseteq V(G^2) - D$ which covers all the vertices in G^2. Clearly, D' forms a minimal inverse dominating set G^2. Since for any graph G, there exists atleast one vertex $v \in V(G)$, such that $\deg(v) = \delta(G)$, it follows that, $|D'| \cup |D_t| \leq |S_1| + \deg(v)$. Therefore, $\gamma^{-1}(G^2) + \gamma_t(G^2) \leq \gamma_t(G) + \delta(G)$.

Theorem 2.2: Let D and D' be, γ-set and γ^{-1}-set of G^2 respectively. If $\gamma(G^2) = \gamma^{-1}(G^2)$, then each vertex in D is of maximum degree.

Proof: For $p = 2$, the result is obvious. Let $p \geq 3$. Since, $V(G) = V(G^2)$ such that G^2 doesnot contain any end vertex, let $D = \{v_1, v_2, \dots, v_n\}$ be a dominating set of G^2. If there exists a vertex $v \in D$ such that v is adjacent to some vertices in $V(G^2) - D$. Then every vertex $w \in D - \{v\}$ is an end vertex in $< D >$. Further, if w is adjacent to a vertex

$u \in V(G^2) - D$. Then $D' = D - \{v, w\} \cup \{u\}$ is a dominating set of G^2, a contradiction. Hence each vertex in D is of maximum degree in G^2.

Theorem 2.3: For any connected graph G, $\gamma^{-1}(G^2) = 1$ if and only if G^2 has at least two vertices of degree $(p-1)$.

Proof: To prove this result, we consider the following two cases.

Case 1. Suppose G^2 has exactly one vertex v of $\deg(v) = p-1$. Then in this case $D = \{v\}$ is a γ-set of G^2. Clearly, $V - D = V - \{v\}$. Further, if $D_1 = \{u\} \epsilon N(v)$ in $V(G^2) - D$, $\deg(u) \leq p-2$ in G^2. Then there exists at least one vertex $w \notin N(u)$ in G^2 such that $D' = D_1 \cup \{w\}$ forms an inverse dominating set of G^2, a contradiction.

Case 2. Suppose G^2 has at least two vertices u and v of $\deg(u) = p-1 = \deg(v)$ such that u and v are not adjacent. Then $D = \{u\}$ dominates G^2 since $\deg(u) = p-1$ and $V(G^2) - D = V(G^2) - \{u\}$. Further, since u and v are not adjacent, $D' = \{v\} \cup V'$ where $V' \subseteq V(G^2) - D$ forms a γ^{-1}-set of G^2, a contradiction.

Conversely, suppose $\deg(u) = p-1 = \deg(v)$, such that u and v are adjacent to all the vertices in G^2. $D' = \{v\} \epsilon N(u)$, where $\{v\} \subseteq V(G^2) - D$ and vice-versa. In any case, we obtain $|D'| = 1$. Therefore $\gamma^{-1}(G^2) = 1$.

Proposition 2.1: $\gamma^{-1}(K_p^2) = \gamma^{-1}(W_p^2) = \gamma^{-1}(K_{1,p}^2) = \gamma^{-1}(K_{p_1,p_2}^2) = 1$.

Theorem 2.4: For any connected (p,q)-graph G without isolates, then $\gamma(G^2) + \gamma^{-1}(G^2) \leq p$.

Proof : Suppose $D = \{v_1, v_2, \ldots, v_n\} \subseteq V(G^2)$ be the γ-set of G^2, the $D' = \{v_1, v_2, \ldots, v_n\} \subseteq V(G^2) - D$ forms a minimal inverse dominating set of G^2. Since $|D| \leq \lceil p/4 \rceil$ and $|D'| \leq \lceil p/3 \rceil$, it follows that, $|D| \cup |D'| \leq p$. Therefore, $\gamma(G^2) + \gamma^{-1}(G^2) \leq p$.

Suppose $V - D$ is not independent, then there exists at least one vertex $u \in D'$such that $N(u) \subseteq V - D$. Clearly, $|D'| = |\{v - D\} - \{u\}|$ and hence, $|D| \cup |D'| \leq p$, a contradiction.

Conversely, if $V - D$ is independent. Then in this case, $|D'| = |V - D|$ in G^2. Clearly, it follows that $|D| \cup |D'| = p$. Hence, $\gamma(G^2) + \gamma^{-1}(G^2) = p$.

Theorem 2.5: For any connected (p,q) -graph G with $p \geq 3$ vertices, $\gamma^{-1}(G^2) + \gamma_i(G) \leq p-1$.

Proof: For $p \leq 2$, $\gamma^{-1}(G^2) + \gamma_i(G) \nleq p-1$. Consider $p \geq 3$, let $F = \{v_1, v_2, \ldots, v_m\}$ be the minimum set of vertices such that for every two vertices $u, v \in F, N(u) \cap N(v) \in V(G) - F$. Suppose there exists a vertex $S = \{v_1, v_2, \ldots, v_k\} \subseteq F$ which covers all the vertices in G and if the subgraph $< S >$ is totally disconnected. Then S forms the minimal independent dominating set of G. Now in G^2, since $V(G) = V(G^2)$ and distance between two vertices is at most two in G^2, there exists a vertex set $D = \{v_1, v_2, \ldots, v_j\} \subseteq S$, which forms minimal γ-set of G^2. Then the complementary set $V(G^2) - D$ contains another set D' such that $N(D') = V(G^2)$. Clearly, D' forms an inverse dominating set of G^2 and it follows that $|D'| \cup |S| \leq p-1$. Therefore, $\gamma^{-1}(G^2) + \gamma_i(G) \leq p-1$.

Theorem 2.6: If every non end vertex of a tree is adjacent to at least one end vertex, then , $\gamma^{-1}(T^2) \leq \left\lceil \frac{p-m}{2} \right\rceil + 1$.

Proof: Let $F = \{v_1, v_2, \ldots, v_m\}$ be the set of all end vertices in T such that vertex $|F| = m$ and $F_1 \in N(F)$. Suppose, vertex $D \subseteq F_1$ is a γ-set of G^2. Then $D' \subseteq V(G^2) - D - F$ forms a minimal inverse dominating set of G^2. Since every tree T contains at least one non end vertex, it follows that $|D'| \leq \left\lceil \frac{p-m}{2} \right\rceil + 1$. Therefore, $\gamma^{-1}(T^2) \leq \left\lceil \frac{p-m}{2} \right\rceil + 1$.

Theorem 2.7: For any connected (p,q)-graph G, $\gamma^{-1}(G^2) + \gamma_c(G) \leq p + \gamma(G)$.

Proof: For $P \leq 5$, the result follows immediately. Let $P \geq 6$, suppose $D = \{v_1, v_2, \ldots, v_n\}, \deg(v_i) \geq 2, 1 \leq i \leq n$ be a minimal dominating set of G. Now we construct a connected dominating set D_c from D by adding in every step at most two components of D forms a connected component in D. Thus we get a connected dominating set D_c after at most $D - 1$ steps. Now in G^2, $V(G) = V(G^2)$. Suppose $D_1 \subseteq D$ be minimal γ-set of G^2. Then there exists a vertex set $D' = \{v_1, v_2, \ldots, v_k\} \subseteq V(G^2) - D_1$, such that $dist(u, v) \geq 2$, for all $u, v \in D'$, which covers all the vertices in G^2. Clearly, D' forms a minimal inverse dominating set of G^2. Hence it follows that $|D'| \cup |D_c| \leq p \cup |D|$. Therefore, $\gamma^{-1}(G^2) + \gamma_c(G) \leq p + \gamma(G)$.

Theorem 2.8: For any connected (p, q)-graph G, $\gamma^{-1}(G^2) + \gamma_c(G^2) \leq \gamma(G^2) + \beta_0(G)$. Equality holds for K_p.

Proof: Let $B = \{v_1, v_2, \ldots, v_k\}$ be the maximum set of vertices such that $< B >$ is totally disconnected and $|B| =$ *β0G.* Now in *G2*, let *D=v1,v2,…,vn⊆VG2* be the minimum set of vertices such that $N(D) = V(G^2)$. Clearly, D forms a minimal γ-set of G^2. Suppose the subgraph $< D >$ is connected, then D itself is a connected dominating set of G^2. Otherwise, construct the connected dominating set D_c from D by adding at most two vertices $u, v \notin D$ between the vertices of D such that $N(D_c) = V(G^2)$ and $< D_c >$ is connected. Further, there exists a vertex set $D' = \{u_1, u_2, \ldots, u_n\} \subseteq V(G^2) - D$ which covers all the vertices in G^2. Clearly, D' forms an inverse dominating set of G^2 with respect to γ-set of G^2. Since $diam(u, v) \geq 2, \forall u, v \in V(G^2)$, it follows that $|D'| \cup |D_c| \leq |D| \cup |B|$. Therefore, $\gamma^{-1}(G^2) + \gamma_c(G^2) \leq \gamma(G^2) + \beta_0(G)$.

Suppose $G \cong K_p$. Then in this case, $|B| = |D| = |D_c| = |D'| = 1$. Therefore, $\gamma^{-1}(G^2) + \gamma_c(G^2) = \gamma(G^2) + \beta_0(G)$.

Theorem 2.9: For any connected (p, q)-graph G, $\gamma^{-1}(G^2) \leq \left\lceil \frac{p - \gamma_e(G)}{2} \right\rceil + 3$.

Proof: Let $F = \{v_1, v_2, \ldots, v_n\}$ be the set of all end vertices in G and let $v_1 \notin N[F]$ in G. Suppose $H \subseteq V_1$ is a dominating set of the subgraph $< V_1 >$. Then $F \cup H$ forms a minimal end dominating set of G. In $G^2, V(G) = V(G^2)$. Suppose $\text{diam}(G) \leq 3$, then $D' = \{u, v\}$ is a minimal γ^{-1}-set of G^2 with respect to γ-set D of G^2, and the result follows immediately. Suppose $\text{diam}(G) \geq 4$, then $D' = \{u_1, u_2, \ldots, u_k\} \subseteq V(G^2) - D - F$ is a minimal inverse dominating set of G^2 with respect to the dominating set D of G^2. Clearly, it follows that $|D'| \leq \left\lceil \frac{p - \{F \cup H\}}{2} \right\rceil + 3$. Therefore, $\gamma^{-1}(G^2) \leq \left\lceil \frac{p - \gamma_e(G)}{2} \right\rceil + 3$.

Theorem 2.10: For any connected (p, q)-graph G, $\gamma^{-1}(G^2) + \gamma_{re}(G^2) \leq p - \alpha_0(G) + 1$. Equality holds for K_p, P_6.

Proof: Let $C = \{v_1, v_2, \ldots, v_i\}$ be the minimal set of vertices which covers all the edges in G such that $|C| = \alpha_0(G)$. Now in G^2, obtain H, the induced subgraph of G^2, i.e., $H = < G^2 >$ such that H contains at least one end vertex $v \in H$. Let $F = \{v_1, v_2, \ldots, v_k\}$ be the set of all such end vertices in G^2. Suppose $V' = V(G^2) - F$ and $I \subseteq V'$, such that diam $(u, v) \geq 2$, for all $u \in I$ and $v \in F$. Then $D = F \cup I'$, where $I' \subseteq I$, covers all the vertices in G^2 such that every vertex not in D is adjacent to a vertex in D and to a vertex in $V(G^2) - D$. Clearly, D forms a minimal restrained dominating set of G^2. Let $D' = \{w_1, w_2, \ldots, w_n\} \subseteq V(G^2)$ forms a minimal γ^{-1}-set of G^2 with respect to dominating set of G^2. Clearly, it follows that $|D'| \cup |D| \leq p - |C| + 1$. Therefore, $\gamma^{-1}(G^2) + \gamma_{re}(G^2) \leq p - \alpha_0(G) + 1$.

For equality, suppose $G \cong K_p$. Then in this case, $|D| = 1 = |D'|$ and $|C| = p - 1$. Therefore, $|D'| \cup |D| = p - |C| + 1$ and hence $\gamma^{-1}(G^2) + \gamma_{re}(G^2) = p - \alpha_0(G) + 1$.

Suppose $G \cong P_6$. Then in this case, $|D| = 2 = |D'|$ and $|C| = \frac{p}{2}$. Clearly, it follows that, $\gamma^{-1}(G^2) + \gamma_{re}(G^2) = p - \alpha_0(G) + 1$.

Theorem 2.11: For any connected (p,q)-graph $G, \gamma^{-1}(G^2)+\gamma_d(G^2) \le 2p-q+1$. Equality holds for $K_{p,}\ p \ge 4$.

Proof: Let $I=\{v_1,v_2,\dots,v_n\} \subseteq V(G^2)$ and let $I' \in N(I)$, for every vertex v$\in I$ in G^2. Now we form the vertex set $X \subseteq V(G^2)$ by random and independent choice of the vertices of I. Define $X_0=\{I \in X: |N(I) \cap X| = \emptyset\}, Y_0 = \{I \notin X: |N(I) \cap X| = \emptyset\}$ and $Y_1 = \{I \notin X: |N(I) \cap X| = I\}$. Further, if $X_0{}' = \bigcup_{I \in X_0} \{I'\}$ and $Y_0{}' = \bigcup_{I \in Y_0} \{I'\}$. Then $|X_0'| \le |X_0|$ and $|Y_0'| \le |Y_0|$. Clearly, the set $D_d = X \cup X0' \cup Y0 \cup Y0' \cup Y1$ forms a γd-set of $G2$. Let $D'=u1,u2,\dots,uk$ be the minimal γ^{-1}-set of G^2 with respect to dominating set D of G^2. It follows that $|D'| \cup |D_d| \le 2p-q+1$ and hence $\gamma^{-1}(G^2)+\gamma_d(G^2) \le 2p-q+1$.

For equality, suppose $G \cong K_p$, with $p \ge 4$ vertices. Then in this case, $|D_d| = 2 = 2(1) = 2\ |D'|$ and q $= \frac{\mathrm{p(p-1)}}{2}$ for K_p. Clearly, $|D'| \cup |D_d| = 2p-q+1$. Hence $\gamma^{-1}(G^2)+\gamma_d(G^2) = 2p-q+1$.

Theorem 2.12: For any connected $(p,q)-$graph G, $\gamma^{-1}(G^2)+diam\,(G) \le p+\gamma(G)-1$.

Proof: Suppose $G \cong K_2$, then obviously, $\gamma^{-1}(G^2)+diam\,(G) = p+\gamma(G)-1$. Let $V(G)$ contains atleast two vertices u and v such that $dist(u,v)$ forms a diametral path in G. Clearly, $dist(u,v) = diam(G)$. Let $F_1=\{v_1,v_2,\dots,v_k\} \subseteq V(G)$ be the set of vertices which are adjacent to all end vertices in G. Suppose $J=\{u_1,u_2,\dots,u_n\}$ be the set of vertices such that $dist(v_i,u_j) \ge 2$, for all $1 \le i \le k$, $1 \le j \le n$. Then $F_1 \cup J'$ where $J' \subseteq J$ forms a minimal γ-set of G. Now in G^2, suppose $D=\{v_1,v_2,\dots,v_m\} \subseteq F_1 \cup J'$ be the minimal γ-set of G^2. Then the complementary set $\mathrm{V(G^2)-D}$ contains the vertex set $D' \subseteq \mathrm{V(G^2)-D}$, which covers all the vertices in G^2. Clearly, D' forms a minimal inverse dominating set of G^2 and it follows that $|D'|+diam\,(G) \le p+|F_1 \cup J'|-1$. Hence $\gamma^{-1}(G^2)+diam\,(G) \le p+\gamma(G)-1$.

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