CONNECTED ROMAN DOMINATION IN GRAPHS

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Abstract

A Roman dominating function on a graph G is a function $f:V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex $u \in V$ for which $f(u)=0$ is adjacent to at least one vertex $v \in V$ for which $f(v)=2$. The weight of a Roman dominating function is the value $f(V)=\sum_{v\in V} f(v)$. The Roman domination number $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function on G. A Roman dominating function on G is connected Roman dominating function of G if either $\langle V_1 \cup V_2\rangle$ or $\langle V_2\rangle$ is connected. The connected Roman domination number $\gamma_{RC}(G)$ of G is the minimum weight of a connected Roman dominating function on G. In this paper we establish the upper bounds, lower bounds and some equality results for $\gamma_{RC}(G)$.

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1. INTRODUCTION

Let $G=(V,E)$ be a simple (p,q) graph with $p=|V|$ and $q=|E|$. We denote open neighborhood of a vertex v of G by $N(v)$ and its closed neighborhood by $N[v]$. For a vertex set $S \subseteq V(G)$, $N(S)=\underset{v\in S}{\cup} N(v)$ and $N[S]=\underset{v\in S}{\cup} N[v]$. The degree of a vertex x denotes the number of neighbors of x in G and $\Delta(G)$ is the maximum degree of G. Also $\delta(G)$ is the minimum degree of G. A set S of vertices in G is a dominating set, if $N[S]=V(G)$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. If S is a subset of $V(G)$, then we denote by $\langle S\rangle$, the subgraph induced by S. A subset S of vertices is independent, if $\langle S\rangle$ has no edge. For notation and graph theory terminology in general we follow [2] or [6].

A Spider is a tree with the property that the removal of all the end paths of length two of T results in an isolated vertex called a head of a spider and the end vertices are called the foot vertices.

Let S be a set of vertices and $u \in S$. We say that a vertex v is a private neighbor of u with respect to S if $N[v] \cap S=\{u\}$. The private neighbor set of u with respect to S is the set $pn[u,S]=\{v; N[v]\cap S=\{u\}\}$.

A Roman dominating function (RDF) on a graph $G=(V,E)$ is a function $f:V \rightarrow \{0,1,2\}$ satisfying the condition that every vertex $u \in V$ for which $f(u)=0$ is adjacent to at least one vertex $v \in V$ for which $f(v)=2$. The weight of a Roman dominating function is the value $f(V)=\sum_{v\in V} f(v)$. The Roman domination number $\gamma_R(G)$ of G is the minimum weight of a Roman dominating function on G. See [4] and [5].

A function $f=(V_0,V_1,V_2)$ is called connected Roman dominating function (CRDF) of G if either $\langle V_1 \cup V_2\rangle$ or

$\langle V_2 \rangle$ is connected. The connected Roman domination number $\gamma_{RC}(G)$ of G is the minimum weight of a CRDF of G.

Independent Roman dominating functions were studied by Adabi et.al in [1]. A Roman dominating function $f=(V_0,V_1,V_2)$ in a graph G is independent RDF or simply IRDF, if $V_1 \cup V_2$ is independent. The independent Roman domination number $i_R(G)$ of G is the minimum weight of an IRDF of G.

In this paper we establish the new concept called connected Roman domination number of G defined by M. H. Muddebihal and Sumangaladevi. The purpose of this paper is to initialize the study of CRDF which gives one of the direct application of minimal CRDF is to win the war. For this we need the continuous flow of communication between the army troops by supply of requirements with minimum cost, by placing weight 1 between the non adjacent vertices of V_1 and V_2 or V_2, which yields a minimal CRDF.

2. RESULTS

Specific values of Connected Roman domination numbers for some class of graphs

In this section we illustrate the connected Roman domination number by determining the value of $\gamma_{RC}(G)$ for several classes of graphs.

Theorem 1:

For the class of paths P_p, cycles C_p, wheels W_p, stars $K_{1,p}$, complete graphs K_p, and complete bipartite graphs $K_{m,n}$. We have

1. $\gamma_{RC}(P_p)=p$ if $p \neq 2$.

   $=\dfrac{p+1}{2}$ if $p=3$.
2. $\gamma_{RC}(C_p)=p-1$ if $p=3$.

   $=p$ if $p \neq 3$.
3. $\gamma_{RC}(W_p)=\delta-1$.
4. $\gamma_{RC}(K_{1,p})=\alpha_0+1$.
5. $\gamma_{RC}(K_p)=\gamma+1$.
6. $\gamma_{RC}(K_{m,n})=2\alpha_0$.

Theorem 2:

Let $f=(V_0,V_1,V_2)$ be a γ_{RC}- function of G. Then V_2 is a γ-set of G if for each vertex $v \in V_1$ is adjacent to at least one vertex of V_2 or the set $V_1=\phi$.

Proof: Let $f=(V_0,V_1,V_2)$ be a γ_{RC} function of G. Further $D=\{v_1,v_2,........,v_n\}, 1 \leq n \leq p$ be the γ-set and $D_C=\{v_1,v_2,......,v_n\}$ where $1 \leq i \leq p$ be the γ_c-set of G. If $V_1 \neq \phi$, then there exists a vertex set $M=\{v_1,v_2,.......,v_j\}$ where $1 \leq j \leq n$ such that $M \in V-D_C$. Hence $|M|=|V_0|$ and $N[M]-(V_0 \cup V_1)=|V_2|$. Now for every vertex set $S=\{u_i;1 \leq i \leq n\}$ and $\{u_i\} \in D_C - V_2$, we have $|S|=|V_1|$. Suppose there exists at least one vertex of $\{w_i;1 \leq i \leq n\} \in V_1$ such that $N(w_i) \notin |V_2|$. Then f is a γ_{RC}- function of G with $\langle V_1 \cup V_2 \rangle$ as a γ-set of G, a contradiction. Hence for each vertex $v \in S$ must be adjacent to at least one vertex of V_2, which gives f as a γ_{RC}-function with V_2 as a γ-set of G. If $V_1=\phi$. Then $|D|=|D_C|$ Hence f is a γ_{RC}-function with V_2 as a γ-set of G.

Theorem 3

For any non-trivial tree T, $\gamma_{RC}(T)=2\gamma(T)$ if and only if every non end vertex of T is adjacent to at least one end vertex.

Proof: Let $H_1=\{v_i;1 \leq i \leq p\}$ and $H_2=\{v_j;1 \leq j \leq p\}$ be the set of non end vertices adjacent to at least one end vertex and the set of non end vertices which are not adjacent to end vertex respectively. Let $f=(V_0,V_1,V_2)$ be a γ_{RC}-function of G. Suppose $H_2 \neq \phi$. Let D and D_C be a γ-set and γ_C-set of G respectively. Then we have following cases.

Case 1: Suppose $H_2=1$ or 2. Then we have two sub cases.

Subcase 1.1: Assume $H_2 = 1$. Let $\{u\} \in H_2$ such that $\{u\} \in N(H_1)$. Then $\{u\} \in V_1$ but $\{u\} \notin D$ which gives that $\gamma_{RC}(T) > 2\gamma(T)$, a contradiction.

Subcase 1.2: Assume $H_2 = 2$ and $\{v_1, v_2\} \in H_2$ such that $\{v_1, v_2\} \in N(H_1)$. Then $\{v_1, v_2\} \in V_1$ but $\{v_1, v_2\} \notin D$, which gives $\gamma_{RC}(T) > 2\gamma(T)$, a contradiction.

Case 2: Suppose $H_2 = k$ and $\{v_k; 3 \le k \le n\} \in H_2$. Then $\forall \{v_l; 1 \le l \le n\} \subseteq \{v_k\}$, $\{v_l\} \in V_1$. But $\{v_{3l}\} \in D$ and $\{v_l - \{v_{3l}\}\} \notin D$ which gives, $\gamma_{RC}(T) > \gamma(T)$ again a contradiction.

For the converse from the above all cases, let $H_2 = \phi$. Then $|v_i| = |V_2| = |D| = |D_C|$. Hence $\gamma_{Rc}(T) = 2|V_2| + |V_1| = 2|D_C| + \phi = 2|D| = 2\gamma(T)$.

Theorem 4:

For any connected graph with $p \ge 3$ vertices, $\gamma_{RC}(G) + \left\lfloor \frac{\gamma(G)}{2} \right\rfloor \le p + 1$.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_{RC}-function of G. Further $D = \{v_i; 1 \le i \le n\}$ and $D_C = \{v_j; 1 \le j \le n\}$ be the γ-set and γ_C-set of G respectively such that $D \subseteq D_C$. Suppose $\{u_i\} \in D$ has at least one private neighbor in $V - D_C$. Then $\{u_i\} \in V_2$ and $(D_C - \{u_i\}) \in V_1$. Suppose there exists $\{v_i\} \subset D$ with no neighbor in $V - D_C$. Then $\{v_i\} \in V_1$

Hence $\gamma_{RC}(G) + \left\lfloor \frac{\gamma(G)}{2} \right\rfloor \le |V_1| + 2|V_2| + 1 \le p + 1$

Theorem 5:

For any graph G, $\gamma(G) \le \gamma_{RC}(G) \le 3\gamma(G)$.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_{RC}-function of G. Since V_2 dominates V_0, the connected induced subgraph of $(V_1 \cup V_2)$ is a CRDF of G.

Therefore, $\gamma(G) \le |V_1 \cup V_2| \le |V_1| + 2|V_2| \le \gamma_{RC}(G)$.

Now we consider the following cases to establish the upper bound for $\gamma_{RC}(G)$. Let D and D_C be a γ-set and γ_C-set respectively in G. Then we have following cases.

Case 1: Suppose G is a tree. Let $S_1 = \{v_1, v_2, ..., v_n\}$ and $S_2 = \{v_1, v_2, ..., v_i\}$, $1 \le i \le n$ be the set of nonend vertices which are adjacent to at least one end vertex and the set of nonend vertices which are not adjacent to end vertices respectively. Then $|S_1| = |V_2|$, $|S_2| = |V_1|$ and $|V_0| = V - (S_1 \cup S_2)$.

Hence $\gamma_{RC}(G) \le |V_1 \cup V_2| \le |V_1| + 2|V_2| = 3|D| \le 3\gamma(G)$.

Case 2: Suppose G is not a tree and $N = \{v_1, v_2, ..., v_n\}$ be the set of all end vertices. Then we have following subcases.

Subcase2.1: Assume $N \ne \phi$, let $H_1 = \{v_1, v_2, ..., v_k\} \subseteq D_C \subset S_2$ be the set nonend vertices with at least one private neighbor in $V - D_C$ and $H_2 \subset D_C \subset S_2$ be the set of nonend vertices with no private neighbor in $V - D_C$. Then $(S_1 \cup H_1) \in V_2$ and $H_2 \in V_1$.

Hence $\gamma_{RC}(G) \le |V_1 \cup V_2| = |V_1| + 2|V_2| \le 3|D| = 3\gamma(G)$.

Subcase2.2: Assume $N = \phi$. Then $S_1 = \phi$ Let $H_1, H_2 \subset D_C \in S_2$ such that H_1 has at least one private neighbor in $V - D_C$ and H_2 has no private neighbor in $V - D_C$. Clearly $H_2 \in V_1$ and $H_1 \in V_2$.

Hence $\gamma_{RC}(G) = |V_1 \cup V_2| = |V_1| + 2|V_2| \le 3|D| = 3\gamma(G)$

Theorem 6:

For any tree T, $\gamma_{RC}(T)=i_R(T)$ if and only if every nonend vertex of T is adjacent to exactly two end vertices or if every nonend vertex of T is adjacent to at least three vertices, then they are not adjacent.

Proof: Suppose there exists at least one nonend vertex of T, which is adjacent to only one end vertex. Let I be the minimal independent Roman dominating set of T and $f=(V_0,V_1,V_2)$ be a γ_{RC}-function of T. Further let $I_1,I_2\subset I$ with $|I_1|=|V_1|$ and $|I_2|=|V_2|$. Then each vertex of $I_2\in V_2$ in $i_R(T)$ and $\gamma_{RC}(T)$. But $I_1\in V_1$ and $N(I_1)\in V_0$ in $i_R(T)$, whereas $N(I_1)\in V_2$ and $I_1\in V_0$ in $\gamma_{RC}(T)$, which gives $\gamma_{RC}(T)>i_R(T)$, a contradiction.

Suppose there exists a nonend vertex set $\{v_i;1\le i\le n\}$ adjacent to at least three end vertices such that at least two vertices of $\{v_i\}$ are adjacent and $\{N\}$ be the set of end vertices of T. Let $A=\{v_i\}$. Then each vertex of $A\in V_2$ in $\gamma_{RC}(T)$. But for the pair of adjacent vertices $(u,v)\subseteq A$ with $\deg u\ge\deg v$, we have $u\in V_2$, $v\in V_0$ and $\{N(v)\cap\{N\}\}\in V_1$ in $i_R(T)$. Since each vertex of A is adjacent to at least three end vertices, $\deg(v)\ge 3$ which gives, $\gamma_{RC}(T)<i_R(T)$, a contradiction.

Conversely, let $f=(V_0,V_1,V_2)$ be a γ_{RC}-function of T and $f^{'}=(V_0^{'},V_1^{'},V_2^{'})$ be a $i_R(T)$- function of T. Assume statement of the Theorem holds. Let I be the minimal independent Roman dominating set of T. Further $\{n_1\}$ and $\{N\}$ be the set of non end vertices and end vertices respectively. Then $\{n_1\}\in V_2^{'}$ or $V_0^{'}$. Let $\{S_1,S_2\}\subset\{n_1\}$ such that $\{S_1\}\in\{V_2^{'}\}$ and $\{S_2\}\in\{V_0^{'}\}$. If $\{S_1\}\in V_2^{'}$. Then $\{S_1\}\in\{V_2\}$. If $\{S_2\}\in\{V_0^{'}\}$. Then for each $\{u_i\}\in\{S_2\}$, there exists exactly two neighbors $(x,y)\in\{N(u_i)\cap\{N\}\}$ such that $(x\cup y)\in V_1^{'}$. Since each $\{u_i\}$ of $\{S_2\}$ has exactly two neighbors (x,y) in $V_1^{'}$, hence there exists $\{w\}\in N(x)\cap N(y)$ such that $\{w\}\in V_2$. Clearly $|V_1\cup V_2|=|V_1^{'}\cup V_2^{'}|$. Hence $\gamma_{RC}(T)=i_R(T)$.

Theorem 7:

Let $G=K_{m_1,m_2,m_3,........,m_n}$ be the complete n-partite graph with $m_1\le m_2\le ....\le m_n$.

a. If $m_1=1$, then $\gamma_{RC}(G)=2$.

b. If $m_2\ge 2$, then $\gamma_{RC}(G)=4$.

Proof: a. This case is obvious.

b. Let M be the partite set of size m_1 and $N=V-M$, further $u\in M$ and $v\in N$ such that $f(u)=f(v)=2$. While every other vertex w is in either M or N, let $f(w)=0$. If there exists a vertex $w\in M\cap V_0$, then there must exists a vertex $x\in N\cap V_2$. If there also exists a vertex $y\in V_0\cap(N-\{x\})$, then there must exists a vertex $z\in\{V_2\cap(M-\{w\})\}$. Since f is an CRDF, we have $\gamma_{RC}(G)=4$.

Theorem 8:

Let T be any tree with $p>3$ vertices. Then $\gamma(T)+\gamma_t(T)\le\gamma_{RC}(T)$

Proof: Let T be any tree with $p>3$ vertices. Suppose $D=\{v_1,v_2,......,v_n\}$ and $D_t=\{v_1,v_2,......,v_i\}$ where $1\le i\le n$ be the γ-set and γ_t-set of G respectively. Then there exists a vertex set $\{u_i\}\in D\subseteq D_t$ of T adjacent to at least one end vertex such that $\{u_i\}\in V_2$ and also there may exists a nonend vertex set $\{v_i\}\in V-(\{u_i\}\cup V_0)$ which are not adjacent to end vertex such that $\{v_i\}\in V_1$.Which gives, $|D|+|D_t|\le|V_1\cup V_2|$. Hence $\gamma(T)+\gamma_t(T)\le\gamma_{RC}(T)$.

Theorem 9:

Let G be any graph with $\gamma(G)=\gamma_C(G)$. Then $\gamma_{RC}(G)=\gamma(G)+\gamma_C(G)$.

Proof: It is sufficient to prove this result for any connected graphs G. Let G be any connected graph with $\gamma(G)=\gamma_C(G)$ and $D=\{v_1,v_2,......,v_n\}$ be the set of vertices of G which forms γ-set for G. Since $\gamma(G)=\gamma_C(G)$. Hence D also forms a connected dominating set of G. Suppose $f=(V_0,V_1,V_2)$ is a CRDF of G. Then $|V_0|=V-D$, $|V_1|=\phi$ and $|D|=|V_2|$. Hence $\gamma_{RC}(G)=|V_1|+2|V_2|=2|D|=|D|+|D|=\gamma(G)+\gamma_C(G)$.

Observation: For any graph G, $\gamma_R(G)\le\gamma_{RC}(G)$.

Theorem 10:

Let G be any connected (p,q) graph. Then $\gamma_{RC}(G)=\gamma_R(G)$ if and only if every vertex $\{v_i\}$ of D_C has at least two private neighbors in $V-D_C$, where D_C is a γ_C-set of G.

Proof: Let $f=\{V_0,V_1,V_2\}$ and $g=\{V_0^{'},V_1^{'},V_2^{'}\}$ be a γ_{RC}-function and γ_R-function of G respectively. Assume $D_C=\{v_1,v_2,.......v_n\}$ be a γ_C-set of G. Then we consider the following cases.

Case 1: Suppose there exists at least one vertex $\{u_i\}$ of D_C with exactly one private neighbor in $V-D_C$. Then for every $\{u_k\}\subseteq\{u_i\}$, we consider following sub cases.

Subcase 1.1: Assume no two vertices of $\{u_k\}$ are adjacent in G. Then $\{u_k\}\in V_0^{'}$ and $\{N(u_k)\cap(V-D_C)\}\in V_1^{'}$. But $\{u_k\}\in V_2$ and $\{N(u_k)\cap(V-D_C)\}\in V_0$, which gives, $\gamma_{RC}(G)>\gamma_R(G)$, a contradiction.

Subcase 1.2: Assume $\forall\{u_k;1\le k\le n\}$, there exists at least two vertices of $\{u_k\}$ which are adjacent in G. Then $\{u_k\}\in V_0^{'}$ or $V_2^{'}$. If $\forall\{u_k\}\in V_0^{'}$. Then there exists $\{v_i\}\in\{N(u_k)\cap(V-D_C)\}$ such that $\forall\{v_i\}\in V_1^{'}$. But $\{u_k\}\in V_2$ and $\{v_i\}\in V_0$, which gives, $\gamma_{RC}(G)>\gamma_R(G)$, a contradiction. If $\forall\{u_k\}\in V_2^{'}$, then there exists at least two vertices of $\{u_j\}\in\{N(u_k)\cap D_C\}$ such that $\{u_j\}\in V_0^{'}$. But for every $\{u_k\cup u_j\}\in V_1$, which gives $\gamma_{RC}(G)>\gamma_R(G)$, a contradiction.

Case 2: Suppose there exists at least one vertex $\{w_i\}$ which has no neighbors in $V-D_C$. Then $\{w_i\}\in V_0^{'}$ or $V_1^{'}$ or $V_2^{'}$. If $\{w_i\}\in V_0^{'}$. Then $\{w_i\}\in V_1$, again $\gamma_{RC}(G)>\gamma_R(G)$, a contradiction. If $\{w_i\}\in V_1^{'}$ or $V_2^{'}$. Then there exists at least two vertices of $\{w_j\}\in N(w_i)$ such that $\{w_j\}\in V_0^{'}$. But $\{w_j\cup N(w_j)\}\in V_1$, which gives $\gamma_{RC}(G)>\gamma_R(G)$, a contradiction.

Hence in all cases, we have $\gamma_{RC}(G)>\gamma_R(G)$, a contradiction.

Conversely, let every vertex $\{v_i\}$ of D_C has at least two private neighbors in $V-D_C$. Then $\{v_i\}\in V_2$ and $\{N(v_i)\cap V-D_C\}\in V_0$, also $\{v_i\}\in V_2^{'}$ and $\{N(v_i)\cap V-D_C\}\in V_0^{'}$.

Hence $\gamma_{RC}(G)=|V_1\cup V_2|=|V_1^{'}\cup V_2^{'}|=\gamma_R(G)$.

We need the following Theorem to prove further Theorem

Theorem A [2]:

For any connected graph G with $\Delta(G)<p-1$, $\gamma(G)\le\gamma_t(G)\le\gamma_C(G)$.

Theorem 11:

Let G be any connected graph with $\Delta(G) < p-1$. Then $\gamma_{RC}(G) \le \gamma_C(G) + \gamma_t(G)$

Proof: Let $f = \{V_0, V_1, V_2\}$ be a γ_{RC}-function of G. Suppose $D_C = \{v_1, v_2, ..., v_n\}$ and $D_t = \{v_1, v_2, ..., v_i\}$, where $1 \le i \le n$ be the γ_C-set and γ_t-set respectively. Then $\gamma_t(G) \le \gamma_C(G)$. We consider the following cases.

Case 1: Suppose $\gamma_t(G) = \gamma_C(G)$. Then $\{D_C\} \in V_2$ or $(V_1 \cup V_2)$ and $\{V - D_C\} \in V_0$. Again we consider the following subcases.

Subcase 1.1: Assume $|D_C| = |V_2|$. Then $V_1 = \phi$ and $|V - D_C| = |V_0|$.

Hence

$$\gamma_{RC}(G) = 2|V_2| = 2|D_C| = |D_C| + |D_t| = \gamma_C(G) + \gamma_t(G)$$

Subcase 1.2: Assume $|D_C| = |V_1 \cup V_2|$.

Then $|V - D_C| = |V_0|$.

Thus

$$\gamma_{RC}(G) = 2|V_2| + |V_1| \le 2\left[|V_2| + |V_1|\right] = 2|D_C| = |D_C| + |D_t| = \gamma_C(G) + \gamma_t(G)$$

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Case 2: Suppose $\gamma_t(G) < \gamma_C(G)$, let $\forall \{v_i\} \in D_t \subset D_C$. Then $\{v_i\} \in V_2$ or V_1 and $D_C - \{v_i\} \in V_1$. Since $\gamma_t(G) < \gamma_C(G)$. Hence there exists at least one vertex $\{v_j\} \notin \{v_i\}$ and $\{v_j\} \in D_C$ such that $\{v_j\} \in V_1$.

Hence

$$\gamma_{RC}(G) = 2|V_2| + |V_1| \le |D_C| + |D_t| = \gamma_C(G) + \gamma_t(G).$$

Theorem 12:

Let $f = (V_0, V_1, V_2)$ be a γ_{RC}-function of G. Then $\gamma_{RC}(G) \le 2\gamma_C(G)$

Proof: We consider a minimal connected dominating set D_C of G. Since each vertex of D_C connected dominating set of G. Hence $D_C \in (V_1 \cup V_2)$. Suppose $V_1 = \phi$. Then $\{D_C\} \in V_2$, which gives $|V_2| = |D_C|$. Clearly $\gamma_{RC}(G) = 2|V_2| = 2|D_C|$. Hence $\gamma_{RC}(G) = 2\gamma_C(G)$. Suppose $V_1 \ne \phi$. Then $|V_1| + |V_2| = |D_C|$. Clearly $\gamma_{RC}(G) = |V_1| + 2|V_2| < 2|V_1| + 2|V_2| = 2\left[|V_1| + |V_2|\right] = 2|D_C|$. Hence $\gamma_{RC}(G) < 2\gamma_C(G)$. Thus $\gamma_{RC}(G) \le 2\gamma_C(G)$

Theorem 13:

Let G be any connected graph. If D_{RC} is a minimal connected Roman dominating function of G and for every $\{v_i\} \in D_{RC}$ there exists at least one vertex of $N(v_i) \in V - D_{RC}$.

Then $V - D_{RC}$ is Roman dominating function of G

Proof: Let D_{RC} be a minimal connected Roman dominating function of G. Suppose for each $\{v_i\} \in D_{RC}$, there exists at least one vertex $\{u_i\} \in N(v_i)$ such that $\{u_i\} \notin V - D_{RC}$. Then $\{u_i\}$ is not dominated by $V - D_{RC}$. Hence $V - D_{RC}$ is not a RDF of G. Thus for each $\{v_i\} \in D_{RC}$, there exists at least one vertex of $N(v_i) \in V - D_{RC}$. Clearly D_{RC} is a minimal connected Roman dominating function and $V - D_{RC}$ is Roman dominating function of G.

Definition: A graph G is said to be Roman connected graph if $\gamma_{RC}(G) = 2\gamma(G)$.

Now we characterize the Roman connected graphs in the following Theorem.

Theorem 14:

A graph G is Roman connected graph if and only if it has a γ_{RC}-function $f = (V_0, V_1, V_2)$ with $|V_1| = \phi$.

Proof: Let G be a graph and $f = (V_0, V_1, V_2)$ be a γ_{RC}-function of G. If $V_1 = \phi$, by definition of $\gamma_{RC}(G)$, V_2

dominates $V-V_2$. Otherwise a connected induced subgraph of $V_1 \cup V_2$ dominates $V-(V_1 \cup V_2)$ and hence $\gamma(G) \le |V_1 \cup V_2| = |V_1| + |V_2| \le |V_1| + 2|V_2| = \gamma_{Rc}(G)$.
We consider the following cases.

Case 1: Suppose $|V_1| \ne \phi$ and D be a γ-set of G. Let $v \in V_1$ and $v \notin D$. Then $2\gamma(G) < 2|V_1 \cup V_2|$, a contradiction.

Case 2: Suppose $|V_1| \ne \phi$ and $v \in D$. Then there exists at least two neighbors of $\{u_i\} \in N(v)$ such that $\{u_i\} \in V_1$ and $\{u_i\} \notin D$, which gives again, $2\gamma(G) < 2|V_1 \cup V_2|$, a contradiction.

Since G is Roman connected. Then $|V_1| = \phi$.

Conversely, let $f = (V_0, V_1, V_2)$ be a γ_{RC}-function of G with $|V_1| = \phi$. Therefore $\gamma_{RC}(G) = 2|V_2|$. Since $|V_1| = \phi$, by Theorem 2, V_2 is a γ-set of G such that $|V_2| = \gamma(G)$. Thus $\gamma_{RC}(G) = 2|V_2| = 2\gamma(G)$. Hence G is Roman connected graph.

Theorem 15:

For any connected graph G with $p \ge 2$ vertices, $\gamma_{RC}(G) = 2\gamma(G)$ if and only if $v \in V$ with degree $p - \gamma(G)$.

Proof: Suppose G has a vertex v with degree $p-\gamma(G)$ such that $v \in V(G)$. If $V_2 = \{v\}$, $V_1 = \phi$ and $V_0 = V - \{v\}$. Then V_2 is a γ-set of G and $f = \{V_0, V_1, V_2\}$ is an CRDF with $f(V) = 2\gamma(G)$. Since $\gamma_{RC}(G) \ge 2\gamma(G)$ for connected graphs, f is a γ_{RC}-function of G.

In order to CRDF $f = \{V_0, V_1, V_2\}$ to have weight $2\gamma(G)$ either **1.** $|V_1| = \gamma(G)+1$ and $|V_2| = \phi$ or **2.** $|V_1| = \phi$ and $|V_2| = \gamma(G)$. Any other arrangement of weight $2\gamma(G)$ would have $|V_1| + |V_2| < 2\gamma(G)$.

For 1, since $|V_2| = \phi$. Then $|V_1| = V$. By a Theorem of Ore [3], $\gamma(G) \le \frac{p}{2}$ for a connected graph G on p vertices. Thus $p = \gamma(G) + 1 \le \frac{p}{2} + 1$, which implies that $p \le 2$. It is easily verified that $\gamma_{RC}(P_2) = 2 = 2\gamma(P_2)$ and P_2 has a vertex of degree 1.

For 2, Let $f = \{V_0, V_1, V_2\}$ be a γ_{RC}-function for G of weight $2\gamma(G)$ with $|V_1| = \phi$ and $|V_2| = \gamma(G)$. Since $V_1 = \phi$, each edge of G joins V_2 and V_2. Hence $\deg(v) = |V_0| = p - |V_1| - |V_2| = p - |V_2| = p - \gamma(G)$.

Theorem 16:

Let T be any tree with every nonend vertex of T is adjacent to at least one end vertex. Then $\gamma_{RC}(T) = 2C$, where C is the set of cut vertices of T.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_{RC}-function of T. Since each nonend vertex of T is adjacent to at least one end vertex. By definition of $\gamma_{RC}(G)$, there exists a connected Roman dominating set $D_{RC} \in V_2$ and $N(D_{RC}) \cap \{N\} \in V_0$ such that $V_1 = \phi$, where C and $\{N\}$ are the set of cut vertices and end vertices of T respectively. Clearly $|C| = |V_2|$. Hence $\gamma_{RC}(T) = 2|V_2| = 2C$.

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