BOUNDS ON DOUBLE DOMINATION IN SQUARES OF GRAPHS

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Abstract

Let the square of a graph G, denoted by G^2 has same vertex set as in G and every two vertices u and v are joined in G^2 if and only if they are joined in G by a path of length one or two. A subset D of vertices of G^2 is a double dominating set if every vertex in G^2 is dominated by at least two vertices of D. The minimum cardinality double dominating set of G^2 is the double domination number, and is denoted by $\gamma_d\left(G^2\right)$. In this paper, many bounds on $\gamma_d\left(G^2\right)$ were obtained in terms of elements of G. Also their relationship with other domination parameters were obtained.

Key words: *Graph, Square graph, Double dominating set, Double domination number.*

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1. INTRODUCTION

In this paper, we follow the notations of [1]. All the graphs considered here are simple, finite and connected. As usual $p=|V(G)|$ and $q=|E(G)|$ denote the number of vertices and edges of G, respectively.

In general, we use $\langle X\rangle$ to denote the subgraph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively. The notation $\alpha_0(G)\left(\alpha_1(G)\right)$ is the minimum number of vertices(edges) is a vertex(edge) cover of G. Also $\beta_0(G)\left(\beta_1(G)\right)$ is the minimum number of vertices (edges) is a maximal independent set of vertex (edge) of G.

Let $\deg(v)$ is the degree of a vertex v and as usual $\delta(G)\left(\Delta(G)\right)$ denote the minimum (maximum) degree of G. A vertex of degree one is called an end vertex and its neighbor is called a support vertex. Suppose a support vertex v is adjacent to at least two end vertices then it is called a strong support vertex. A vertex v is called cut vertex if removing it from G increases the number of components of G.

The distance between two vertices u and v is the length of the shortest uv- path in G. The maximum distance between any two vertices in G is called the diameter, denoted by $diam(G)$.

The square of a graph G, denoted by G^2 has the same vertex set as in G and the two vertices u and v are joined in G^2 if and only if they are joined in G by a path of length one or two (see [1], [2]).

We begin by recalling some standard definitions from domination theory.

A set $S\subseteq V$ is said to be a double dominating set of G, if every vertex of G is dominated by at least two vertices of S. The double domination number of G is denoted by $\gamma_d(G)$ and is the minimum cardinality of a double dominating set of G. This concept was introduced by F. Harary and T. W. Haynes [3].

A dominating set $S\subseteq V(G)$ is a restrained dominating set of G, if every vertex not in S is adjacent to a vertex in S and to a vertex in $V-S$. The restrained domination number of G, denoted by $\gamma_{re}(G)$ is the minimum cardinality of a restrained dominating set of G. This concept was introduced by G. S. Domke et. al.,[4].

A dominating set $S\subseteq V(G)$ is said to be connected dominating set of G, if the subgraph $\langle S\rangle$ is not disconnected. The minimum cardinality of vertices in such a set is called the connected domination number of G and is denoted by $\gamma_c(G)$ [5].

A subset $D\subseteq V\left(G^2\right)$ is said to be a dominating set of G^2, if every vertex not in D is adjacent to some vertex in D. The domination number of G^2, denoted by $\gamma\left(G^2\right)$, is the minimum cardinality of a dominating set of G^2.

Similarly, a dominating set D of G^2 is said to be total dominating set of G^2, if for every vertex $v \in V\left(G^2\right)$, there exists a vertex $u \in D$, $u \neq v$, such that u is adjacent to v or if the subgraph $\langle D \rangle$ has no isolated vertex. The total domination number of G^2, denoted by $\gamma_t(G^2)$ is the minimum cardinality of total dominating set of G^2. Domination parameters in squares of graphs were introduced by M. H. Muddebihal et. al.,(see [6] and [7]).

Analogously, a subset $D \subseteq V\left(G^2\right)$ is said to be double dominating set of G^2, if every vertex in G^2 is dominated by at least two vertices of D. The double domination number of G^2, denoted by $\gamma_d\left(G^2\right)$, is the minimum cardinality of a double dominating set of G^2. In this paper, many bounds on $\gamma_d\left(G^2\right)$ were obtained in terms of elements of G. Also its relationship with other different domination parameters were expressed.

2. RESULTS

Theorem 2.1:

a. For any cycle C_p, with $p \ge 3$ vertices,

$$\gamma_d\left(C_p^2\right) = \begin{cases} 2, \ for \ \ p = 3. \\ \dfrac{p}{3} + 1, \ for \ \ p \equiv 0 (\mathrm{mod}\, 3) \\ \left\lceil \dfrac{p}{3} \right\rceil, \ otherwise. \end{cases}$$

b. For any complete graph K_p, with $p \ge 2$ vertices,

$\gamma_d\left(K_p^2\right) = 2$.

c. For any star $K_{1,n}$, with $n \ge 2$ vertices,

$\gamma_d\left(K_{1,n}^2\right) = 2$.

d. For any wheel W_p, with $p \ge 4$ vertices,

$\gamma_d\left(W_p^2\right) = 2$.

e. For any complete bipartite graph K_{p_1, p_2}, with $p_1 + p_2 = p$ vertices,

$\gamma_d\left(K_{p_1, p_2}^2\right) = 2$.

Theorem 2.2: For any connected graph G with $p \ge 3$ vertices, $\gamma_d\left(G^2\right) \le \left\lceil \frac{p}{2} \right\rceil + 1$.

Proof: For $p \le 2$, $\gamma_d\left(G^2\right) \nleq \left\lceil \frac{p}{2} \right\rceil$. For $p \ge 3$, we prove the result by induction process. Suppose $p = |V| \le 3$ in G, then $\gamma_d\left(G^2\right) = \left\lceil \frac{p}{2} \right\rceil$. Assume that the result is true for any graph with p-vertices. Let G be a graph with $p+1$ vertices. Then by induction hypothesis, it follows that $\gamma_d\left(G^2\right) \le \left\lceil \frac{p+1}{2} \right\rceil$. Hence the result is true for all graphs with $p \ge 3$ vertices by induction process.

Theorem 2.3: For any connected graph G with $p \ge 3$ vertices, $\gamma_d\left(G^2\right) + \gamma\left(G^2\right) \le p$. Equality holds if and only if $G \cong C_3, P_3$.

Proof: Let $S = \{v_1, v_2, ..., v_k\}$ be the minimal set of vertices which covers all the vertices in G^2. Clearly, S forms a dominating set of G^2. Further, if there exists a vertex set $V\left(G^2\right) - S = V_1$ in G^2. Then $S \cup V' = D$, where $V' \subseteq V_1$ in G^2, be the set of vertices such that $\forall v \in V\left(G^2\right)$, there exists two vertices in $S \cup V' = D$. Further, since every vertex of G^2 are adjacent to at least two vertices of G^2, clearly D forms a double dominating set of G^2. Therefore, it follows that $|D| \cup |S| \le p$. Hence $\gamma_d\left(G^2\right) + \gamma\left(G^2\right) \le p$.

Suppose, $G \ncong C_3, P_3$. Then either $2|S| \neq |D|$ or $|D| \cup |S| < p$, which gives a contradiction in both cases.

Suppose, $G \cong C_3, P_3$. Then in this case, $|D| = 2 = 2 \cdot 1 = 2|S|$. Clearly, $|D| \cup |S| = 3 = p$. Therefore, $\gamma_d\left(G^2\right) + \gamma\left(G^2\right) = p$.

Theorem 2.4: For any connected (p,q)- graph G, $2\gamma\left(G^2\right) \le \gamma_d\left(G^2\right) + 2$.

Proof: Suppose $S = \{v_1, v_2, ..., v_n\} \subseteq V\left(G^2\right)$ be the minimal set of vertices which covers all the vertices, such that $dist(u,v) \ge 3$ for all $\{u,v\} \in S$. Then S forms a minimal dominating set of G^2. Further, if for every $v \in V\left(G^2\right)$, there exists at least two vertices $\{u,w\} \in S$ such that $\forall u, v$, $N(v)$ and $N(u)$ belongs to $V\left(G^2\right) - S$. Then S itself is a double dominating set of G^2. Otherwise, there exists at least one vertex $x \in N(S)$ such that $S \cup \{x\} = D$ forms a double dominating set of G^2. Since for any graph G with $p \geq 2$, $\gamma_d\left(G^2\right) \geq 2$. Therefore, it follows that $|S| \leq \frac{|D|+2}{2}$. Clearly, $2\gamma\left(G^2\right) \leq \gamma_d\left(G^2\right)+2$.

Theorem 2.5: For any connected (p,q)-graph G, with $p \geq 3$ vertices, $\gamma_d\left(G^2\right)+\gamma(G) \leq p$. Equality holds for $C_3, C_4, P_3, P_4, P_5, P_7$.

Proof: Let $F_1=\{v_1,v_2,...,v_m\}$ be the set of all non end vertices in G. Suppose $S=\{v_1,v_2,...,v_k\} \subseteq F_1$, $k \leq m$, be the minimal set of vertices which are at distance three covers all the vertices of G. Then S itself forms a minimal γ-set of G. Otherwise, there exists at least one vertex $v \in N[S]$ such that $S \cup \{v\}$ forms a minimal dominating set of G. Now in G^2, since $V(G)=V\left(G^2\right)$, let $I=\{u_1,u_2,...,u_i\}$ be the set of all strong support vertices. Suppose $D=I \cup F_1^{'}$, where $F_1^{'} \subseteq F_1 - I$ be the minimum set of vertices which covers all the vertices in G^2, such that for every vertex $v \in V\left(G^2\right)$, there exists at least two vertices $\{u,w\} \in D$ where $\forall v_i \in I$ and $\forall v_j \in F_1^{'}$ $\exists \{v_k\} \in V[G^2]-D$ has at least two neighbors which are either v_i or v_j. Then D forms a minimal double dominating set of G^2. Therefore, it follows that $|D| \cup |S \cup \{v\}| \leq p$. Hence $\gamma_d\left(G^2\right)+\gamma(G) \leq p$.

Suppose $G \cong C_3, C_4, P_3, P_4$. Then in this case, $|D|=2$ and $|S|=p-2$. Clearly, it follows that $|D| \cup |S| = p$. Therefore, $\gamma_d\left(G^2\right)+\gamma(G)=p$.

Suppose $G \cong P_5, P_7$. Then in this case, $|D|=\left\lceil \frac{p}{2} \right\rceil$ and $|S|=\left\lfloor \frac{p}{2} \right\rfloor$. Clearly, it follows that $|D| \cup |S| = \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{p}{2} \right\rfloor$. Therefore, $\gamma_d\left(G^2\right)+\gamma(G)=p$.

Theorem 2.6: For any connected (p,q)- graph G with $p \geq 3$ vertices, $\gamma_d\left(G^2\right) \leq p - \left\lceil \frac{diam(G)}{2} \right\rceil$.

Proof: For $p=2$, $\gamma_d\left(G^2\right) \not\leq p - \left\lceil \frac{diam(G)}{2} \right\rceil$. Hence consider $p \geq 3$. Suppose there exists two vertices $u,v \in V(G)$, which constitutes the longest path in G. Then $dist(u,v)=diam(G)$. Since $V(G)=V\left(G^2\right)$, there exists a vertex set $D=\{v_1,v_2,...,v_i\}$ such that for every vertex $v_j \in D, 1 \leq j \leq i$, there exists at least one vertex $v_k \in D, 1 \leq k \leq i$. Also every vertex in G^2 is adjacent to at least two vertices of D in G^2. Then D forms a minimal double dominating set of G^2. Since $|D| \geq 2$ and the diametral path includes at least two vertices. It follows that, $2|D| \leq 2p - diam(G)$. Clearly, $\gamma_d\left(G^2\right) \leq p - \left\lceil \frac{diam(G)}{2} \right\rceil$.

Theorem 2.7: For any nontrivial tree with $p \geq 3$ vertices and m cut vertices, then $\gamma_d\left(T^2\right) \leq m+1$, equality holds if $T \cong K_{1,n}$, $n \geq 2$.

Proof: Let $B=\{v_1,v_2,...,v_i\}$ be the set of all cut vertices in T with $|B|=m$. Suppose $A=\{v_1,v_2,...,v_j\}$, $1 \leq j \leq i$ be the set of cut vertices which are at a distance two from the end vertices of T and $A \subset B$. Now in T^2, all the end vertices are adjacent with $\forall v_j \in A$ and $\{B\}-\{A\}$. Now in T^2, since $V(T)=V\left(T^2\right)$, for every vertex $v \in V\left(T^2\right)$, there exists at least two vertices $\{u,v\} \in B = D$ in T^2. Further, since D covers all the vertices in T^2, D itself forms a minimal double dominating set of T^2. Since every tree T contains at least one cut vertex, it follows that $|D| \leq m+1$. Hence $\gamma_d\left(T^2\right) \leq m+1$.

Suppose T is isomorphic to a star $K_{1,n}$. Then in this case, $|D|=2$ and $m=1$. Therefore, it follows that $\gamma_d\left(T^2\right)=m+1$.

Theorem 2.8: For any connected (p,q)-graph G, $\gamma_d\left(G^2\right) \leq \gamma_t\left(G^2\right)+\Delta(G)$.

Proof: For $p=2$, the result follows immediately. Hence, let $p \geq 3$. Suppose $V_1=\{v_1,v_2,...,v_n\} \subseteq V(G)$ be the set of all vertices with $\deg(v_i) \geq 2, 1 \leq i \leq n$. Then there exists at least one vertex $v \in V_1$ of maximum degree $\Delta(G)$. Now in G^2, since $V(G)=V\left(G^2\right)$, let $D_1=\{v_1,v_2,...,v_k\} \subseteq V_1$ in G^2. Suppose D_1 covers all the vertices in G^2 and if the subgraph $\langle D_1 \rangle$ has no isolated vertex, then D_1 itself is a minimal total dominating set of G^2. Otherwise, there exists a set $D_1 \cup H$, where $H \subseteq V\left(G^2\right)-D_1$, forms a minimal total dominating set of G^2. Now let $D=\{v_1,v_2,...,v_j\} \subseteq V_1$ in G^2 be the minimal set of vertices, which covers all the vertices in G^2. Suppose $\forall v \in V\left(G^2\right)$, there exists at least two vertices $\{u,w\}\in D$ which are adjacent to at least one vertex of D and at least two vertices of $V\left(G^2\right)-D$. Then D forms a γ_d- set of G^2. Otherwise $D\cup I$, where $I\subseteq V\left(G^2\right)-D$, forms a minimal double dominating set of G^2. Clearly, it follows that $|D\cup I|\le |D_1\cup H|+\Delta(G)$. Therefore, $\gamma_d\left(G^2\right)\le \gamma_t\left(G^2\right)+\Delta(G)$.

Theorem 2.9: For any connected (p,q)-graph G, $\gamma(G)\le \gamma_d\left(G^2\right)$. Equality holds if and only if $\gamma(G)=2$ with $diam(G)\le 3$.

Proof: If $V_1=\{v_1,v_2,...,v_n\}\subseteq V(G)$ be the set of vertices with $\deg(v_i)\ge 2, 1\le i\le n$. Then $S=\{v_1,v_2,...,v_k\}\subseteq V_1$ forms a minimal dominating set of G. Now without loss of generality in G^2, since $V(G)=V\left(G^2\right)$. If $V_2=\{v_1,v_2,...,v_k\}$ be the set of vertices with $\deg(v_k)<2$. If $V_2\in V(G)$, then the vertices which are at a distance at least two are adjacent to each vertex of V_2 in G^2. Hence $S_1\cup V_2=D$ where $S_1\subseteq S$ forms a minimal double dominating set of G^2. If $V_2=\phi$, then $S\cup V_3=D$ where $V_3\subseteq V_1$ forms a minimal double dominating set of G^2. Further, since every vertex in G^2 is adjacent to atleast two vertices of D, it follows that $|S|\le|D|$. Hence, $\gamma(G)\le \gamma_d\left(G^2\right)$.

Suppose $\gamma(G)\ne 2$ with $diam(G)\le 3$. Then in this case $diam(G)=1$ and hence, $|S|=1$. Clearly, $|S|<|D|$. Therefore $\gamma(G)<\gamma_d\left(G^2\right)$, a contradiction.

Further, if $\gamma(G)=2$ with $diam(G)\nleq 3$. Then in this case, $diam(G)\ge 4$. Clearly, $|D|>|S|$. Therefore, $\gamma(G)<\gamma_d\left(G^2\right)$, again a contradiction.

Hence $\gamma(G)=\gamma_d\left(G^2\right)$ if and only if $\gamma(G)=2$ with $diam(G)\le 3$.

Theorem 2.10: For any connected (p,q)-graph G, $\gamma_d\left(G^2\right)\le p-\alpha_0(G)+1$. Equality holds for K_p.

Proof: Let $A=\{v_1,v_2,...,v_n\}\subseteq V(G)$, where $\deg(v_i)\ge 2, 1\le i\le n$, be the minimum set of vertices which covers all the edges of G, such that $|A|=\alpha_0(G)$. Now in G^2 since $V(G)=V\left(G^2\right)$, let $D=\{v_1,v_2,...,v_k\}\subseteq A$ be the set of vertices such that for every vertex $v\in V\left(G^2\right)$, there exists at least two vertices $\{u,w\}\in D$ in G^2. Further, if D covers all the vertices in G^2, then D itself is a double dominating set of G. Clearly, it follows that $|D|\le p-|A|+1$ and hence $\gamma_d\left(G^2\right)\le p-\alpha_0(G)+1$.

Suppose $G\cong K_p$. Then in this case, $|A|=p-1$ and $|D|=2$. Clearly, it follows that $|D|=p-|A|+1$ and hence $\gamma_d\left(G^2\right)=p-\alpha_0(G)+1$.

Theorem 2.11: For any connected (p,q)-graph G, $\gamma_d\left(G^2\right)\le \gamma_t(G)$.

Proof: Let $K=\{u_1,u_2,...,u_n\}\subseteq V(G)$ be the set of vertices such that $N[u_i]\cap N\left[u_j\right]=\phi$, where $1\le i\le n$, $1\le j\le n$. Suppose there exists a minimal set $K_1=\{u_1,u_2,...,u_k\}\in N(K)$, such that the subgraph $\langle K\cup K_1\rangle$ has no isolated vertex. Further, if $K\cup K_1$ covers all the vertices in G, then $K\cup K_1$ forms a minimal total dominating set of G. Since $V(G)=V\left(G^2\right)$, there exists a vertex set $D=\{v_1,v_2,...,v_m\}\subseteq K\cup K_1$ in G^2, which covers all the vertices in G^2 and for every vertex $v\in V\left(G^2\right)$, there exists at least two vertices $\{u,w\}\in D$. Clearly, D forms a minimal double dominating set of G^2. Therefore, it follows that $|D|\le|K\cup K_1|$. Hence $\gamma_d\left(G^2\right)\le \gamma_t(G)$.

Theorem 2.12: For any connected (p,q)-graph G, $\gamma_d\left(G^2\right)\le \beta_0(G)+1$. Equality holds for K_p.

Proof: For $p=2$, the result is obvious. Hence let $p\ge 3$. Suppose $F=\{u_1,u_2,...,u_m\}\subseteq V(G)$ be the set of all vertices with $\deg(v_i)=1, 1\le i\le m$. Then $F\cup F^{'}$, where $F^{'}\subseteq V(G)-F$, $F^{'}\notin N[F]$ forms a maximal independent set of vertices, such that $\left|F\cup F^{'}\right|=\beta_0(G)$. Since $V(G)=V\left(G^2\right)$, let $D_1=\{v_1,v_2,...,v_n\}\subseteq V\left(G^2\right)-F$ and $D_1\in N(F)$. Suppose $D_2\subseteq V\left(G^2\right)-D_1$ such that $D_1\cup D_2=D$ forms a minimal set of vertices which covers all the vertices in G^2. Further, if for every vertex $v\in V\left(G^2\right)$, there exists at least two vertices $\{u,w\}\in D$. Then D forms a minimal double dominating set of G^2. Since every graph G contains at least one independent vertex, it follows that $|D|\le\left|F\cup F^{'}\right|+1$. Therefore, $\gamma_d\left(G^2\right)\le \beta_0(G)+1$.

Suppose $G \cong K_p$. Then in this case, G contains exactly one independent vertex and by Theorem 2.1(b), it follows that $\gamma_d\left(G^2\right) = \beta_0\left(G\right)+1$.

Theorem 2.13: For any non-trivial tree T, $\gamma_d\left(T^2\right) \le \gamma_{re}\left(T\right)+1$.

Proof: Let $F=\{v_1,v_2,...,v_n\}\subseteq V\left(T\right)$ be the set of vertices with $\deg\left(v_i\right)=1, \forall\{v_i\}\in F, 1\le i\le n$. Suppose for every vertex $v\in V\left(T\right)-F$, there exists a vertex $u\in F$ and also a vertex $x\in V\left(T\right)-F$. Then F itself is a restrained dominating set of T. Otherwise, there exists at least one vertex $w\in V\left(G\right)-F$, such that $D^{'}=F\cup\{w\}$ forms a minimal restrained dominating set of T. Let $D=\{u_1,u_2,...,u_k\}\subseteq V-F$ in T^2 be the minimal set of vertices which are chosen such that $\forall v\in V\left(T^2\right)$, there exists at least two vertices $\{y,z\}\in D$. Further, since D covers all the vertices in T^2, clearly D forms a minimal double dominating set of T^2. Therefore, it follows that $\left|D\right|\le\left|D^{'}\right|+1$ due to the distance between vertices of T is one. Hence $\gamma_d\left(T^2\right)\le\gamma_{re}\left(T\right)+1$.

Theorem 2.14: For any connected graph G, $\gamma_d\left(G^2\right)\le\gamma_c\left(G\right)+1$.

Proof: Suppose $C=\{v_1,v_2,...,v_n\}\subseteq V\left(G\right)$ be the set of all cut vertices in G. Further, if $C\cup I$, where $I\in N\left(C\right)$ with $\deg\left(v_i\right)\ge 2, \forall\{v_i\}\in I$ be the minimal set of vertices which covers all the vertices in G and if the sub graph $\langle C\cup I\rangle$ is connected. Then $C\cup I$ forms a minimal connected dominating set of G. Let $D=\{v_1,v_2,...,v_k\}$ be the minimal set of vertices which covers all the vertices in G^2. Suppose for every vertex $v\in V\left(G^2\right)$, there exists at least two vertices $\{u,w\}\in D$. Then D itself forms a minimal double dominating set of G^2. Therefore, it follows that $\left|D\right|\le\left|C\cup I\right|+1$ and hence $\gamma_d\left(G^2\right)\le\gamma_c\left(G\right)+1$.

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