ON GENERATING FUNCTIONS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

B. C. Chandra[1], K. P. Samanta[2], C. S. Bera[3]

[1]Sree Vidyaniketan, 9A, J. M. Avenue, Kolkata – 700006
[2, 3]Department of Mathematics, Bengal Engineering and Science University, Shibpur,
P.O. Botanic Garden, Howrah - 711 103, India
biplab_chandra72@yahoo.com, kalipadasamanta2010@gmail.com, chandrasekharbera75@gmail.com

Abstract

In this paper, we have obtained some novel generating functions (both bilateral and mixed trilateral) involving modified bi-orthogonal polynomials $Y_{n+m}^{\alpha+n}(x;k)$, by group-theoretic method. As particular cases, we obtain the corresponding results on generalized Laguerre polynomials.

Key words: *AMS-2000 Classification Code :33C45,33C47, Biorthogonal polynomials, Laguerre polynomials, generating functions.*

* * *

1. INTRODUCTION

An explicit representation for the Konhauser bi-orthogonal polynomials [1], $Y_n^\alpha(x;k)$, suggested by the Laguerre polynomials was given by carlitz [2] in the following form:

$$Y_n^\alpha(x;k)=\frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

Where, $(a)_n$ is the pochhammer symbol, $\alpha>-1$ and k is a positive integer.

The aim at presenting this paper is to obtain some results (new and known) on bilateral and mixed trilateral generating functions for the modified Konhauser biorthogonal polynomials $Y_{n+m}^{\alpha+n}(x;k)$, by the group-theoretic method .Several particular cases of interest are also discussed in this paper .The main results of our investigation are stated in the form of the following theorems :

Theorem 1:

If there exists a unilateral generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty}a_n\,Y_{n+m}^{\alpha+n}(\mathrm{x;k})\,w^n,$$

Then

$$(1-kw)^{\frac{-(1+\alpha+mk)}{k}}\exp\left(x\left[1-(1-kw)^{\frac{-1}{k}}\right]\right)G\big(x(1-kw),\quad tw(1-kw)\big)$$

$$=\sum_{n=0}^{\infty}w^n\,\sigma_n(x,t),$$

Where

$$\sigma_n(x,t)=\sum_{p=0}^{n}a_p\,k^{n-p}\binom{n+m}{n-p}Y_{n+m}^{\alpha+p}(\mathrm{x;k})\,t^p.$$

Theorem 2:

If there exists a generating relation of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n\,Y_{n+m}^{\alpha+n}(\mathrm{x;k})\mathrm{g_n(u)}\,w^n$$

Where $\mathrm{g_n(u)}$ is an arbitrary polynomial of degree n and right hand series have formal power series expansion, then

$$(1-kw)^{\frac{-(1+\alpha+mk)}{k}}\exp\left(x\left[1-(1-kw)^{\frac{-1}{k}}\right]\right)G(x(1-kw),u,\quad vw(1-kw))$$

$$=\sum_{n=0}^{\infty}w^n\,\sigma_n(x,u,v),$$

Where

$$\sigma_n(x,u,v)=\sum_{r=0}^{n}a_r\,k^{n-r}\binom{n+m}{n-r}Y_{n+m}^{\alpha+r}(\mathrm{x;k})\mathrm{g_r(u)}\,v^r.$$

2. DERIVATION OF A NEW GENERATING FUNCTION:

At first we consider the following operator R:

$$R = x\,y^{-1}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (k+1)z^2y^{-1}\frac{\partial}{\partial z} + (1-x+km)y^{-1}z \qquad (2.1)$$

Such that

$$R(Y_{n+m}^{\alpha+n}(\mathrm{x};\mathrm{k})y^{\alpha}z^{n}) = k(n+m+1)Y_{n+m+1}^{\alpha+n}(\mathrm{x};\mathrm{k})y^{\alpha-1}z^{n+1}. \qquad (2.2)$$

The extended form of the group generated by R is given as follows:

$$e^{wR}\,f(x,y,z) = (1-kwy^{-1}z)^{-\frac{(km+1)}{k}}\exp\left(x - \frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)$$

$$\times f\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}},\frac{y}{(1-kwy^{-1}z)^{\frac{1}{k}}},\frac{z}{(1-kwy^{-1}z)^{1+\frac{1}{k}}}\right), (2.3)$$

Where $f(x,y,z)$ is an arbitrary function and w is an arbitrary constant. Now using (2.3), we obtain

$$e^{wR}(Y_{n+m}^{\alpha+n}(x;k)y^{\alpha}z^{n}) = (1-kwy^{-1}z)^{-\frac{(km+kn+\alpha+n+1)}{k}}y^{\alpha}z^{n} \times \exp\left(x - \frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)Y_{n+m}^{\alpha+n}\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}};k\right). \qquad (2.4)$$

But using (2.2), we obtain

$$e^{wR}(Y_{n+m}^{\alpha+n}(x;k)y^{\alpha}z^{n}) = \sum_{p=0}^{\infty}\frac{w^p}{p\,!}R^p(Y_{n+m}^{\alpha+n}(x;k)y^{\alpha}z^{n})$$

$$= \sum_{p=0}^{\infty}\binom{n+m+p}{n+m}Y_{n+m+p}^{\alpha+n}(x;k)y^{\alpha}z^{n}(kwy^{-1}z)^{p}\ . \qquad (2.5)$$

Equating (2.4) and (2.5) and then putting $kwy^{-1}z = w$, we get

$$(1-w)^{-\frac{(1+\alpha+n+nk+mk)}{k}}\exp\left(x - \frac{x}{(1-w)^{\frac{1}{k}}}\right)Y_{n+m}^{\alpha+n}\left(\frac{x}{(1-w)^{\frac{1}{k}}};k\right) = \sum_{p=0}^{\infty}\binom{n+m+p}{n+m}Y_{n+m+p}^{\alpha+n}(x;k)w^{p}. \qquad (2.6)$$

Now putting m=0 in (2.6), we get

$$(1-w)^{-\frac{(1+\alpha+n+nk)}{k}}\exp\left(x - \frac{x}{(1-w)^{\frac{1}{k}}}\right)Y_{n}^{\alpha+n}\left(\frac{x}{(1-w)^{\frac{1}{k}}};k\right) = \sum_{p=0}^{\infty}\binom{n+p}{n}Y_{n+p}^{\alpha+n}(x;k)w^{p}, \qquad (2.7)$$

Which does not seem to have appeared in the earlier works. Now putting n=0 and replacing α by $\alpha-n$ in (2.6), we get

$$(1-w)^{-\frac{(km+\alpha+1)}{k}}\exp\left(x - \frac{x}{(1-w)^{\frac{1}{k}}}\right)Y_{m}^{\alpha}\left(\frac{x}{(1-w)^{\frac{1}{k}}};k\right) = \sum_{p=0}^{\infty}\binom{m+p}{m}Y_{m+p}^{\alpha}(x;k)w^{p}, \qquad (2.8)$$

Which is found derived in [3] by the classical method. Now putting m=0 in (2.8),we get

$$(1-w)^{-\frac{(1+\alpha)}{k}}\exp\left(x - \frac{x}{(1-w)^{\frac{1}{k}}}\right) = \sum_{p=0}^{\infty}Y_{p}^{\alpha}(x;k)w^{p}, \qquad (2.9)$$

Which is found derived in [3,4].

Special Case 1:

If we put $k=1$, then $Y_n^{\alpha}(x;k)$ reduce to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$.Thus Putting $k=1$ in the above generating relations, we get the following generating relation on Laguerre polynomials:

$$(1-w)^{-(1+\alpha+2n)}\exp\left(\frac{-xw}{1-w}\right)L_n^{(\alpha+n)}\left(\frac{x}{1-w}\right)=\sum_{p=0}^{\infty}\binom{n+p}{p}L_{n+p}^{(\alpha+p)}(x)w^p \quad (2.10)$$

$$(1-w)^{-(1+\alpha+m)}\exp\left(\frac{-xw}{1-w}\right)L_m^{(\alpha)}\left(\frac{x}{1-w}\right)=\sum_{p=0}^{\infty}\binom{m+p}{p}L_{m+p}^{(\alpha)}(x)w^p \quad (2.11)$$

And

$$(1-w)^{-(1+\alpha)}\exp\left(\frac{-xw}{1-w}\right)=\sum_{p=0}^{\infty}L_p^{(\alpha)}(x)w^p. \quad (2.12)$$

The generating functions (2.11) and (2.12) are found derived in [5, 6, 7].

3. PROOF OF THEOREM 1

Let us consider the generating relation of the form:

$$G(x,w)=\sum_{n=0}^{\infty}a_nY_{n+m}^{\alpha+n}(x;k)\,w^n \quad (3.1)$$

Replacing w by twz and multiplying both sides of (3.1) by y^α and finally operating e^{wR} on both sides, we get

$$e^{wR}\left(y^\alpha\,G(x,twz)\right)=e^{wR}\left(\sum_{n=0}^{\infty}a_n(Y_{n+m}^{\alpha+n}(x;k)y^\alpha z^n)(tw)^n\right). (3.2)$$

Now the left member of (3.2), with the help of (2.3), reduces to

$$(1-kwy^{-1}z)^{-\frac{(\alpha+km+1)}{k}}\exp\left(x-\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)y^\alpha\times G\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}},\quad\frac{twz}{(1-kwy^{-1}z)^{1+\frac{1}{k}}}\right). \quad 3.3)$$

The right member of (3.2), with the help of (2.2), becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{n}a_n\frac{w^p}{p!}k^p\,(n+m+1)_pY_{n+m+p}^{\alpha+n}(x;k)y^{\alpha-p}z^{n+p}(tw)^n$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n}a_{n-p}\frac{w^n}{p!}k^p\,(n-p+m+1)_pY_{n+m}^{\alpha+n-p}(x;k)y^{\alpha-p}z^n(t)^{n-p}(3.4)$$

Now equating (3.3) and (3.4) and then substituting $y=z=1$, we get

$$(1-kw)^{-\frac{(1+mk+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kw)^{\frac{1}{k}}}\right)G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},\frac{wt}{(1-kw)^{1+\frac{1}{k}}}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n}a_{n-p}\frac{w^n}{p!}k^p\,(n-p+m+1)_pY_{n+m}^{\alpha+n-p}(x;k)t^{n-p}.$$

Therefore

$$(1-kw)^{-\frac{(1+mk+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kw)^{\frac{1}{k}}}\right)G\left(\frac{x}{(1-kw)^{\frac{1}{k}}},\frac{wt}{(1-kw)^{1+\frac{1}{k}}}\right)$$

$$=\sum_{n=0}^{\infty}w^n\,\sigma_n(x,t),$$

Where

$$\sigma_n(x,t)=\sum_{p=0}^{n}a_pk^{n-p}\binom{n+m}{n-p}Y_{n+m}^{\alpha+n-p}(x;k)t^p.$$

This completes the proof of the theorem which is believed to be the new one.

4. PROOF OF THEOREM 2:

Again Let us consider the another generating relation of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nY_{n+m}^{\alpha+n}(x;k)\,g_n(u)\,w^n\,. \quad (4.1)$$

Replacing $w$by $wvz$and then multiplying both sides of (4.1) by y^α and finally applying the operator e^{wR} on both sides, we easily obtain, as in section 3, the following generating relation:

$$(1-kwy^{-1}z)^{-\frac{(1+mk+\alpha)}{k}}\exp\left(x-\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}\right)y^\alpha$$

$$\times G\left(\frac{x}{(1-kwy^{-1}z)^{\frac{1}{k}}}, u, \frac{wvz}{(1-kwy^{-1}z)^{1+\frac{1}{k}}}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} a_{n-r}\frac{w^n}{r!}k^r\,(n-r+m+1)_r Y_{n+m}^{\alpha+n-r}(x;k)\,g_{n-r}(u)\,y^{\alpha-r}z^n v^{n-r}$$

Now substituting $y = z = 1$, we get

$$(1-kw)^{-\frac{(1+mk+\alpha)}{k}}\exp\left(x - \frac{x}{(1-kw)^{\frac{1}{k}}}\right) \times G\left(\frac{x}{(1-kw)^{\frac{1}{k}}}, u, \frac{wv}{(1-kw)^{1+\frac{1}{k}}}\right)$$

$$= \sum_{n=0}^{\infty} w^n\,\sigma_n(x,u,v),$$

Where

$$\sigma_n(x,u,v) = \sum_{r=0}^{n} a_r k^{n-r}\binom{n+m}{n-r} Y_{n+m}^{\alpha+r}(x;k)\,g_r(u)v^r.$$

This completes the proof of the theorem which is believed to be the new one.

Special Case 2:

If we put $k = 1$ in our Theorem 1 & 2, we get the following result on generalized Laguerre polynomials:

Theorem 3:

If there exists a generating relation of the form:

$$G(x,w) = \sum_{n=0}^{\infty} a_n L_{n+m}^{(\alpha+n)}(\mathrm{x})\,w^n,$$

Then

$$(1-w)^{-(1+\alpha+m)}\exp\left(\frac{-wx}{1-w}\right) G\left(\frac{x}{1-w}, \qquad \frac{tw}{(1-w)^2}\right) = \sum_{n=0}^{\infty} w^n\,\sigma_n(x,t),$$

Where

$$\sigma_n(x,t) = \sum_{p=0}^{n} a_p\binom{n+m}{n-p} L_{n+m}^{(\alpha+p)}(\mathrm{x})\,t^p,$$

Which does not seem to have appeared in the earlier works.

Theorem 4:

If there exists a generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_{n+m}^{(\alpha+n)}(\mathrm{x})\,g_n(u)\,w^n,$$

Then

$$(1-w)^{-(1+\alpha+m)}\exp\left(\frac{-wx}{1-w}\right) \times G\left(\frac{x}{1-w}, u, \frac{tw}{(1-w)^2}\right) = \sum_{n=0}^{\infty} w^n\,\sigma_n(x,u,v),$$

Where

$$\sigma_n(x,u,v) = \sum_{r=0}^{n} a_r\binom{n+m}{n-r} L_{n+m}^{(\alpha+r)}(\mathrm{x})\,g_r(u)\,v^r,$$

Which does not seem to have appeared in the earlier works.

REFERENCES

[1] Konhauser, J.D.E., Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J.Math., 21(1967), 303-314.
[2] Carlitz, L.,A note on certain biorthogonal polynomials, Pacific J.Math.,24(1968),425-430.
[3] Prabhakar, T.R., On the other set of biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J.Math., 37(1971), 801-804.
[4] Srivastava, H.M., Some Biorthogonal polynomials suggested by the Laguerre polynomials, Pacific J.Math.,98(1982)235-250.
[5] E.D.Rainville, Special Functions, Macmillan, New York, 1960.
[6] Truesdell, C., An essay toward a unified theory of special functions, Princeton university press, 1948.
[7] McBride, E.B., Obtaining Generating Functions, Springer Verlag, Berlin, 1972.