NUMERICAL SOLUTION OF BOUSSINESQ EQUATION ARISING IN ONE-DIMENSIONAL INFILTRATION PHENOMENON BY USING FINITE DIFFERENCE METHOD

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Abstract

Infiltration is a gradual flow or movement of groundwater into and through the pores of an unsaturated porous medium (soil). The fluid infiltered in porous medium (unsaturated soil), its velocity decreases as soil becomes saturated, and such phenomena is called infiltration. The present model deals with the filtration of an incompressible fluid (typically, water) through a porous stratum, the main problem in groundwater infiltration. The present model was developed first by Boussinesq in 1903 and is related to original motivation of Darcy. The mathematical formulation of the infiltration phenomenon leads to a non-linear Boussinesq equation. In the present paper, a numerical solution of Boussinesq equation has been obtained by using finite difference method. The numerical results for a specific set of initial and boundary conditions are obtained for determining the height of the free surface or water mound. The moment infiltrated water enters in unsaturated soil; the infiltered water will start developing a curve between saturated porous medium and unsaturated porous medium, which is called water table or water mound. Crank-Nicolson finite difference scheme has been applied to obtain the required results for various values of time. The obtained numerical results resemble well with the physical phenomena. When water is infiltered through the vertical permeable wall in unsaturated porous medium the height of the free surface steadily and uniformly decreases due to the saturation of infiltered water as time increases. Forward finite difference scheme is conditionally stable. In the present paper, the graphical representation shows that Crank-Nicolson finite difference scheme is unconditionally stable. Numerical solution of the governing equation and graphical presentation has been obtained by using MATLAB coding.

Key words: *Infiltration, porous media, Darcy's law, Crank-Nicolson finite difference scheme*

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1. INTRODUCTION

The groundwater flow plays an important role in various fields like Agriculture, fluid dynamics, Chemical engineering, Environmental problems, Biomathematics and nuclear waste disposal problems. The infiltration phenomenon is useful to control salinity of water, contamination of water and agriculture purpose. Such problems are also useful to measure moisture content of water in vertical one-dimensional ground water recharge and dispersion of any fluid in porous media. Infiltration is a gradual flow or movement of groundwater into and through the pores of an unsaturated porous medium (soil). Infiltration is governed by two forces, gravity and capillary action. While smaller pores offer greater resistance to gravity, very small pores pull water through capillary action in addition to and even against the force of gravity. V´azquez reported that the present model was developed first by Boussinesq in 1903 and is related to original motivation of Darcy [15]. It has been discussed by number of prominent authors from different viewpoints; for example, by Darcy (1856), Jacob Bear (1946), M. Muskat (1946), A.E.Scheidegier (1960) and Polubarinova-Kochina P Ya (1962). In addition, the focus has also been thrown on the groundwater infiltration phenomenon, especially, in homogeneous porous media as well as heterogeneous porous media by Verma(1967), Mehta & Verma(1977), M.N.Mehta(2006), Mehta & Patel(2007), Mehta & Yadav(2007), Mehta & Joshi (2009), Mehta & Meher(2010), Mehta & Desai (2010) and Mehta, Pradhan & Parikh(2011)from various aspects and viewpoints.

The present model deals with the filtration of an incompressible fluid (typically, water) through a porous stratum, the main problem in groundwater infiltration. According to Polubarinova-Kochina and Scheidegger AE the moment infiltrated water enters in unsaturated soil; the infiltered water will start developing a curve between saturated porous medium and unsaturated porous medium, which is called water table or water mound [10, 13]. In this investigated model an attempt has been made to measure the height of the free surface of water mound.

2. STATEMENT OF THE PROBLEM

Consider reservoir field with water of height $h_m = 1 =$ maximum height with impermeable bottom and surrounding of this reservoir is unsaturated homogeneous soil. To understand infiltration phenomenon in uni-direction, a vertical cross section area of the reservoir with surrounding unsaturated porous medium is considered, as shown in the following figure 1.

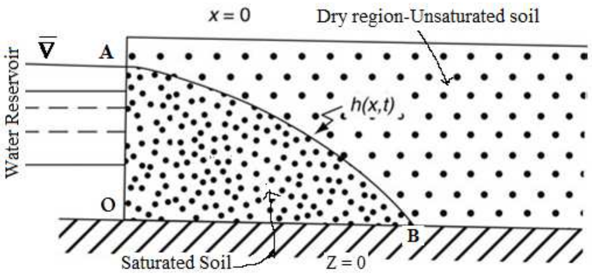

Fig-1: A schema of groundwater infiltration.

The infiltration is a process by which the water of the reservoir has entered into the unsaturated soil through vertical permeable wall. The infiltrated water will enter in unsaturated soil then the infiltered water will develop a curve between saturated porous medium and unsaturated porous medium, which is called water table or water mound. To measure the height of the free surface is the basic purpose of the investigation. To understand this one-dimensional infiltration phenomenon and for the sake of its mathematical formulation, some assumptions have been taken. The governing equation for the height of infiltered water is obtained in the form of a non-linear partial differential equation known as Boussinesq's equation [2]. The scheme for the solution of Boussinesq's equation has been suggested by Klute, Rosenberg and Smith using Finite Difference Methods [5, 11, 12]. Bear explained that atmospheric pressure in dry region by using relation between pressure and height of free surface and velocity of infiltered water can be calculated by Darcy's law [1, 3, and 9].

3. MATHEMATICAL FORMULATION

The height of water in the reservoir is assumed as $OA = 1 =$ Maximum height of free surface by $h(x,t)$ (figure 1). The height of the free surface is 0, when $OB = x = 1$. The dotted arc below the curve is saturated by infiltered water and above the curve is dry region of unsaturated porous medium. The water is infiltered through the height OA, the bottom is assumed impermeable, so water can not flow in downward direction. To develop and understand the mathematical formulation of the infiltration phenomenon it is necessary to impose (consider) the following simplifying assumptions:

The stratum has height $h_m = 1$ and lies on the top of a horizontal impervious bed, which is labelled as $Z = 0$. Ignore the transversal variable y; and the water mass which infiltrates the soil occupies a region described as

$$\Omega = \{(x, z) \in R: \ z \le h(x,t)\}$$

In practical terms, it is assumed that there is no region of partial saturation. This is an evolution model. Clearly, $0 \le h(x,t) \le 1$, as $h_m = 1$ is the maximum height and the free boundary surface $h(x,t)$ is also an unknown of the problem. For the sake of simplicity and for the practical computation after introducing suitable assumptions, the hypothesis of almost horizontal flow, it is assumed that the flow has an almost horizontal speed. Here, the y-component of the velocity of infiltered water will be zero. Here, $u \approx (u, 0)$ so that $h(x,t)$ has small gradients. It follows that in the vertical component, the momentum equation will be,[15]

$$\rho\left(\frac{\partial u_z}{\partial t} + \mathrm{u}\cdot\nabla u_z\right) = -\frac{\partial p}{\partial z} - \rho g \qquad (1)$$

Neglecting the inertial term (the left-hand side), integration in z gives for this first approximation $p + \rho g z = \text{constant}.$ now calculate the constant on the free surface $z = h(x,t)$.

If continuity of the pressure across the interface is imposed, then $\rho = 0$ (assuming constant atmospheric pressure in the air that fills the pores of the dry region $z > h(x,t)$.) this implies

$$p = \rho g\,(h - z) \qquad (2)$$

In other words, the pressure is determined by means of the hydrostatic approximation. Now, using mass conservation law and taking a section $S = (x, x+a) \times (0, C)$

$$\phi \frac{\partial}{\partial t}\int_x^{x+a}\int_0^h dydx = -\int_{\partial s} \mathrm{u}\cdot n\, dl \qquad (3)$$

Where ϕ is the porosity of the medium, i.e., the fraction of volume available for the flow circulation, and u is the velocity, which obeys Darcy's law in the form that includes gravity effects

$$\mathrm{u} = -\frac{\mathrm{k}}{\mu}\nabla\left(p+\rho g z\right) \qquad (4)$$

On the right-hand lateral surface it is,

$$\mathrm{u}\cdot\mathrm{n} \approx \left(\mathrm{u},0\right)\cdot\left(1,0\right) = \mathrm{u},\ \text{i.e. } -\left(\frac{k}{\mu}\right)px,$$

While on the left-hand side it is, -u.

Using formula for p and differentiating in x it follows that,

$$\phi\frac{\partial h}{\partial t} = \frac{\rho g k}{\mu}\frac{\partial}{\partial x}\int_0^h \frac{\partial}{\partial x} h dz \qquad (5)$$

Thus, the governing equation of the phenomenon known as Boussinesq's equation, is obtained as

$$\frac{\partial h}{\partial t} = \beta\frac{\partial^2}{\partial x^2}\left(h^2\right) \qquad (6)$$

Where constant $\beta = \frac{\rho g k}{2\phi\mu}$ and h has a small horizontal gradient.

From the expression (6) it follows that

$$\frac{\partial h}{\partial t} = \beta\frac{\partial}{\partial x}\left(\frac{\partial h^2}{\partial x}\right)$$

$$\text{i.e. } \frac{\partial h}{\partial t} = 2\beta\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)$$

\On simplifying, it gives

$$\frac{\partial h}{\partial t} = 2\beta\left(h\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2\right) \qquad (7)$$

The expression (7) is a nonlinear partial differential equation known as Boussinesq equation, which is required governing equation of the height of free surface of infiltered water in unsaturated porous medium, Vazquez [15].

4. NUMERICAL SOLUTION OF THE PROBLEM

To obtain the numerical solution of the equation (7), choose dimensionless variable

$X = \frac{x}{L}$ and $T = 2\beta t$ as $0 \le X \le 1$, $0 \le T \le 1$

Hence, the equation (7) can be written as

$$\frac{\partial h}{\partial T} = h\frac{\partial^2 h}{\partial X^2} + \left(\frac{\partial h}{\partial X}\right)^2 \qquad (8)$$

The appropriate initial condition to the phenomena may be consider as

$h\left(X,0\right) = 1 - X^2,\quad 0 \le X \le 1$

and boundary condition according to the figure-1will be (9)

$h\left(0,T\right) = 1,\qquad X = 0,\ \mathrm{T} > 0$

$h\left(1,T\right) = 0,\qquad \mathrm{T} > 0$

As per Rosenberg, the Crank-Nicolson finite difference scheme is employed to solve equation (8) with the conditions (9) as follows [11]:

Hence, the stability ratio $r = \frac{\Delta T}{\left(\Delta X\right)^2}$

Choosing the above scheme for $i = 1$,

$$\left[-2h_{1,n+\frac{1}{2}} - \frac{2}{r}\right]h_{1,n+1} + \left[h_{1,n+\frac{1}{2}} + \frac{1}{4}\left(h_{2,n+\frac{1}{2}} + h_{1,n+\frac{1}{2}} - 4\right)\right]h_{2,n+1}$$
$$= \left[2h_{1,n+\frac{1}{2}} - \frac{2}{r}\right]h_{1,n} - \left[h_{1,n+\frac{1}{2}} + \frac{1}{4}\left(h_{2,n+\frac{1}{2}} + h_{1,n+\frac{1}{2}} - 4\right)\right]h_{2,n}$$
$$-2\left[h_{1,n+\frac{1}{2}} - \frac{1}{4}\left(h_{2,n+\frac{1}{2}} + h_{1,n+\frac{1}{2}} - 4\right)\right] \qquad (10)$$

With

$$h_{1,n+\frac{1}{2}} = h_{1,n} + \frac{r}{2}\left[h_{1,n}\cdot\left(h_{2,n} - 3h_{1,n} + 2\right) + \frac{1}{4}\left(h_{2,n} + h_{1,n} - 2\right)^2\right]$$

For $2 \le i \le R-1$,

$$\left[h_{i,n+\frac{1}{2}} - \frac{1}{2}\left(\frac{h_{i+1,n+\frac{1}{2}} - h_{i-1,n+\frac{1}{2}}}{2}\right)\right]h_{i-1,n+1} + \left[-2h_{i,n+\frac{1}{2}} - \frac{2}{r}\right]h_{i,n+1}$$
$$+\left[h_{i,n+\frac{1}{2}} + \frac{1}{2}\left(\frac{h_{i+1,n+\frac{1}{2}} - h_{i-1,n+\frac{1}{2}}}{2}\right)\right]h_{i+1,n+1}$$
$$= -\left[h_{i,n+\frac{1}{2}} - \frac{1}{2}\left(\frac{h_{i+1,n+\frac{1}{2}} - h_{i-1,n+\frac{1}{2}}}{2}\right)\right]h_{i-1,n} + \left[2h_{i,n+\frac{1}{2}} - \frac{2}{r}\right]h_{i,n}$$
$$-\left[h_{i,n+\frac{1}{2}} + \frac{1}{2}\left(\frac{h_{i+1,n+\frac{1}{2}} - h_{i-1,n+\frac{1}{2}}}{2}\right)\right]h_{i+1,n} \qquad (11)$$

With

$$h_{i,n+\frac{1}{2}} = h_{i,n} + \frac{r}{2}\left[h_{i,n}\cdot\left(h_{i+1,n} - 2h_{i,n} + h_{i-1,n}\right) + \frac{1}{4}\left(h_{i+1,n} - h_{i-1,n}\right)^2 \right]$$

For $i = R$,

$$\left[h_{R,n+\frac{1}{2}} + \frac{1}{4}\left(h_{R-1,n+\frac{1}{2}} + h_{R,n+\frac{1}{2}} \right) \right] h_{R-1,n+1} + \left[-2h_{R,n+\frac{1}{2}} - \frac{2}{r} \right] h_{R,n+1} = -\left[h_{R,n+\frac{1}{2}} + \frac{1}{4}\left(h_{R-1,n+\frac{1}{2}} + h_{R,n+\frac{1}{2}} \right) \right] h_{R-1,n} + \left[2h_{R,n+\frac{1}{2}} - \frac{2}{r} \right] h_{R,n} \qquad (12)$$

With

$$h_{R,n+\frac{1}{2}} = h_{R,n} + \frac{r}{2}\left[h_{R,n}\cdot\left(h_{R-1,n} - 3h_{R,n}\right) + \frac{1}{4}\left(h_{R-1,n} + h_{R,n}\right)^2 \right]$$

The expressions (10), (11) and (12) represent the Crank-Nicolson finite difference scheme about the point $\left(X_i, T_{i,n+\frac{1}{2}}\right)$ for the infiltration phenomenon represented by the expression (8). Mehta, Pradhan and others have discussed the solution of such phenomena [4, 6, 7, 8, 9] analytically and numerically with different view point.

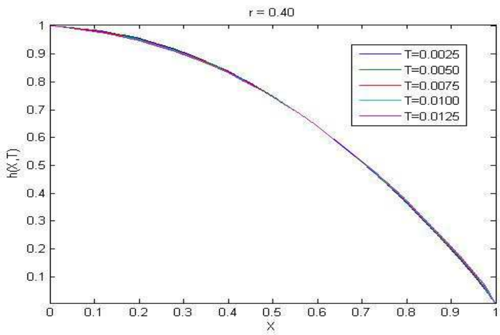

Fig-2: Graph of $h(X,T)$ at different times for $r = 0.40$

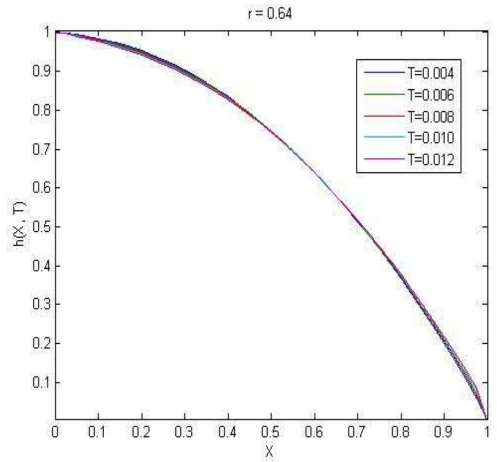

Fig-3: Graph of $h(X,T)$ at different times for $r = 0.64$

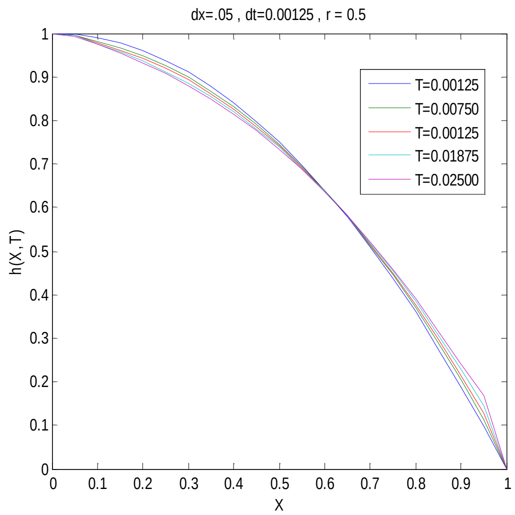

Fig- 4: Graph of $h(X,T)$ at different times for $r = 0.5$

5. NUMERICAL AND GRAPHICAL PRESENTATION

Table-1: Values of $h(X,T)$ at times T for r = 0.64

	dX=0.025 dT=0.0004 r=0.64				
X	T=0.004	T=0.006	T=0.008	T=0.010	T=0.012
0	1	1	1	1	1

0.025	0.998565	0.998317	0.998115	0.997943	0.997793
0.05	0.994075	0.993117	0.992327	0.991651	0.991058
0.075	0.989236	0.987629	0.986291	0.985136	
0.1	0.983813	0.981656	0.979831	0.978242	0.976833
0.125	0.977611	0.975024	0.972792	0.970827	0.969070
0.15	0.970479	0.967585	0.965036	0.962762	0.960712
0.175	0.962306	0.959222	0.956447	0.953937	0.951652
0.2	0.953017	0.949842	0.946928	0.944254	0.941795
0.225	0.942565	0.939379	0.936402	0.933634	0.931061
0.25	0.930922	0.927787	0.924812	0.922011	0.919382
0.275	0.918074	0.915034	0.912115	0.909335	0.906701
0.3	0.904011	0.901101	0.898279	0.895567	0.892976
0.325	0.888731	0.885976	0.883286	0.880680	0.878171
0.35	0.872231	0.869651	0.867120	0.864652	0.862260
0.375	0.854512	0.852124	0.849773	0.847470	0.845225
0.4	0.835571	0.833392	0.831241	0.829124	0.827051
0.425	0.815410	0.813455	0.811520	0.809609	0.807729
0.45	0.794029	0.792311	0.790607	0.788919	0.787252
0.475	0.771426	0.769961	0.768503	0.767054	0.765617
0.5	0.747603	0.746405	0.745207	0.744011	0.742819
0.525	0.722560	0.721642	0.720718	0.719790	0.718858
0.55	0.696295	0.695672	0.695037	0.694390	0.693732
0.575	0.668810	0.668496	0.668163	0.667811	0.667442
0.6	0.640104	0.640113	0.640096	0.640052	0.639985
0.625	0.610178	0.610524	0.610836	0.611115	0.611362
0.65	0.579030	0.579728	0.580384	0.580998	0.581574
0.675	0.546663	0.547726	0.548738	0.549702	0.550619
0.7	0.513074	0.514517	0.515900	0.517227	0.518497
0.725	0.478265	0.480101	0.481869	0.483571	0.485208
0.75	0.442234	0.444479	0.446645	0.448735	0.450749
0.775	0.404984	0.40765	0.410227	0.412715	0.415118
0.8	0.366512	0.369613	0.372612	0.375509	0.378306
0.825	0.326819	0.330366	0.333794	0.337104	0.340301
0.85	0.285902	0.2899	0.293757	0.297477	0.301074
0.875	0.243754	0.248194	0.252463	0.256581	0.260577
0.9	0.20034	0.205176	0.209816	0.214313	0.218727
0.925	0.155527	0.160643	0.165591	0.170483	0.17542
0.95	0.108809	0.114024	0.119302	0.124805	0.13068
0.975	0.058347	0.063827	0.070094	0.077279	0.085543
1	0	0	0	0	0

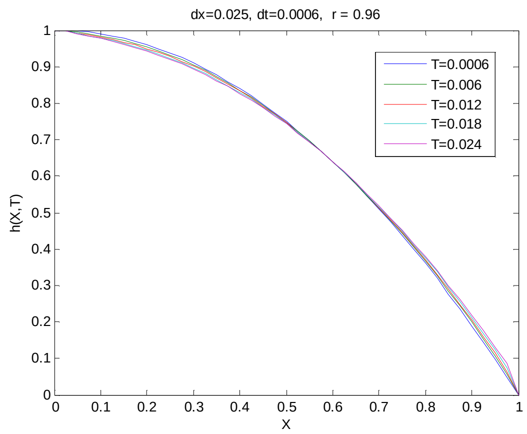

Fig-5: Graph of $h(X,T)$ at different times for $r = 0.96$

Table-2: Values of $h(X,T)$ at times T for r = 0.4

dX= 0.025 dT= 0.00025 r = 0.40					
X/T	T=0.0025	T=0.0050	T=0.0075	T=0.0100	T=0.0125
0	1	1	1	1	1
0.025	0.998803	0.998601	0.998434	0.99829	0.998162
0.05	0.994979	0.994213	0.99357	0.993011	0.992513
0.075	0.990720	0.989464	0.988393	0.987451	0.986607
0.1	0.985752	0.984115	0.982686	0.981414	0.980264
0.125	0.979867	0.977967	0.976267	0.974729	0.973324
0.15	0.972925	0.970869	0.968985	0.967250	0.965646
0.175	0.964839	0.962715	0.960723	0.958859	0.957115
0.2	0.955562	0.953432	0.951397	0.949464	0.947633
0.225	0.945071	0.942976	0.940947	0.938996	0.937127
0.25	0.933356	0.931324	0.929336	0.927406	0.925540
0.275	0.920411	0.918460	0.916541	0.914662	0.912832
0.3	0.906236	0.904380	0.902547	0.900743	0.898975
0.325	0.890829	0.889079	0.887346	0.885635	0.883951
0.35	0.874192	0.872557	0.870936	0.869332	0.867747
0.375	0.856322	0.854812	0.853314	0.851828	0.850357
0.4	0.837221	0.835846	0.834479	0.833122	0.831776
0.425	0.816889	0.815656	0.814431	0.813212	0.812002
0.45	0.795325	0.794244	0.793169	0.792098	0.791032
0.475	0.772529	0.771610	0.770693	0.769779	0.768867
0.5	0.748502	0.747753	0.747004	0.746255	0.745507
0.525	0.723244	0.722674	0.722102	0.721527	0.720950
0.55	0.696754	0.696372	0.695986	0.695594	0.695197
0.575	0.669032	0.668848	0.668656	0.668456	0.668248
0.6	0.640079	0.640101	0.640112	0.640113	0.640103
0.625	0.609894	0.610132	0.610356	0.610565	0.610761
0.65	0.578478	0.578940	0.579385	0.579813	0.580224
0.675	0.545831	0.546526	0.547201	0.547855	0.548490

0.7	0.511951	0.512889	0.513803	0.514693	0.515560
0.725	0.476841	0.478030	0.479192	0.480326	0.481434
0.75	0.440498	0.441948	0.443367	0.444754	0.446111
0.775	0.402925	0.404644	0.406328	0.407977	0.409591
0.8	0.364119	0.366117	0.368075	0.369994	0.371872
0.825	0.324082	0.326368	0.328607	0.330801	0.332948
0.85	0.282813	0.285393	0.287919	0.290390	0.292806
0.875	0.240311	0.243187	0.245996	0.248736	0.251410
0.9	0.196563	0.199719	0.202784	0.205764	0.208669
0.925	0.151509	0.154868	0.158110	0.161263	0.164357
0.95	0.104808	0.108145	0.111409	0.114660	0.117945
0.975	0.054672	0.057680	0.060951	0.064509	0.068381
1	0	0	0	0	0

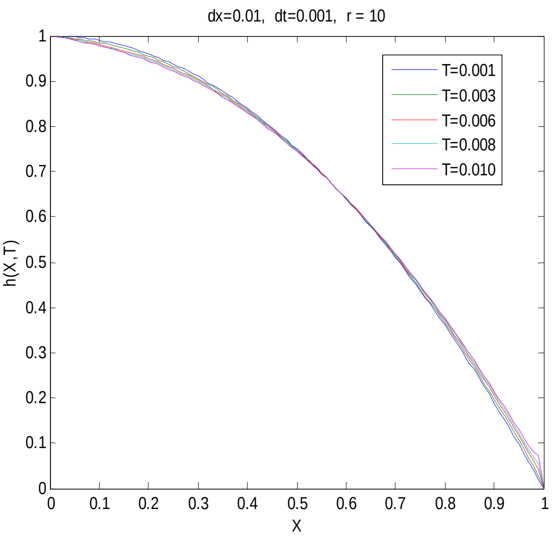

Fig-6: Graph of $h(X,T)$ at different times for $r=10$

CONCLUSIONS

The Numerical solution of the groundwater infiltration phenomenon given by the expression (8) has been obtained by using Crank-Nicolson finite difference scheme. The initial and boundary conditions given in the expression (9) have been used.

The graphs show that height of infiltered water mound or height of free surface of infiltered water in unsaturated porous medium is decreasing for different time $T>0$. The nature of the curves (figures 2, 3, 4, 5, 6 and tables 1 and 2) reflect that the height of free surface of infiltered water in unsaturated porous medium is decreasing according to the physical phenomenon throughout the domain

For the given distance X for different times T, $T=0.010, T=0.012$, $T=0.018$, $T=0.024$ etc., and the solution converges to 0 as X tending to 1.Thus, when water is infiltered through the vertical permeable wall in unsaturated porous medium the height of the free surface steadily and uniformly decreases due to the saturation of infiltered water as time increases. The numerical results are shown. (figures 2, 3, 4, 5, 6) for the distinct values of the stability ratio, $r = 0.4, 0.5, 0.64, 0.96, 10;$. In the forward finite difference scheme numerical solution is stable only for $r = \Delta T / (\Delta X)^2 \le 0.5$, this restriction has been overcome by employing Crank-Nicolson finite difference scheme.

Hence, we conclude that the Crank-Nicolson finite difference scheme gives numerical solution of the non-linear equation arising in infiltration phenomenon which is consistent with the physical phenomenon which is stable without having any stringent restriction on the stability ratio.

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