CONTROLLER DESIGN OF INVERTED PENDULUM USING POLE PLACEMENT AND LQR

P. Kumar[1], O.N. Mehrotra[2], J. Mahto[3]

Abstract

In this paper modeling of an inverted pendulum is done using Euler – Lagrange energy equation for stabilization of the pendulum. The controller gain is evaluated through state feedback and Linear Quadratic optimal regulator controller techniques and also the results for both the controller are compared. The SFB controller is designed by Pole-Placement technique. An advantage of Quadratic Control method over the pole-placement techniques is that the former provides a systematic way of computing the state feedback control gain matrix.LQR controller is designed by the selection on choosing. The proposed system extends classical inverted pendulum by incorporating two moving masses. The motion of two masses that slide along the horizontal plane is controllable .The results of computer simulation for the system with Linear Quardatic Regulator (LQR) & State Feedback Controllers are shown in section 6.

Keyword-*Inverted pendulum, Mathematical modeling Linear-quadratic regulator, Response, State Feedback controller, gain formulae.*

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1. INTRODUCTION

One of the most celebrated and well – publicized problems in control system is Inverted Pendulum[3,4] or broom balancer problem. This is an unstable system[1,17]] that may model a rocket before launch. Almost all known and novel control techniques have been tested on IP problem.This is a classical problem in dynamics and control theory[2,5] and is widely used as a benchmark[16] for testing control algorithms(PID controllers, Linear Quadratic Regulator (LQR), neural networks, fuzzy logic control, genetic algorithms, etc)[7,8].The inverted pendulum is unstable[11] in the sense that it may fall over any time in any direction unless a suitable control force is applied. The control objective of the inverted pendulum is to swing up[4] the pendulum hinged on the moving cart by a linear motor[12] from stable position (vertically down state) to the zero state(vertically upward state)[6,9] and to keep the pendulum in vertically upward state in spite of the disturbance[10,13].It is highly nonlinear[12,15], but it can be easily be controlled by using linear controllers in an almost vertical position[18]. If the system is controllable or at leasr stabilizable, this method gives excellent stability margins. The guaranteed margins in LQR design are 60 degree phase margin, infinite gain margin ,and -6dB gain reduction margin.

2. MATHEMATICAL MODEL OF PHYSICAL SYSTEM

The inverted pendulum is a classical problem in dynamics and control theory and is widely used as a benchmark for testing control algorithms.

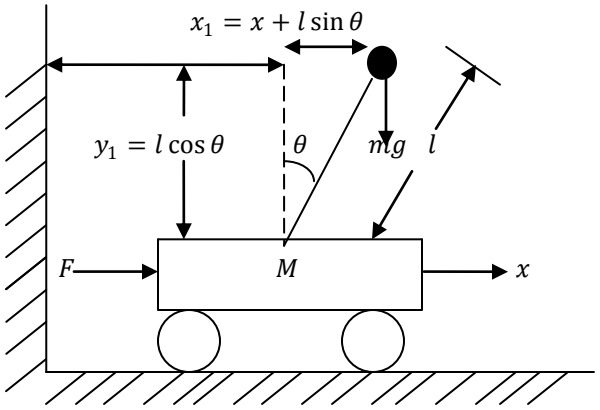


Fig 1 : The Inverted Pendulum System

Let the new ordinate of the centre of gravity of the pendulum be (x_1, y_1).

Define the angle of the rod from the vertical (reference) line as θ and displacement of the cart as x. Also assume the force applied to the system is F , g be the acceleration due to gravity and l be the half length of the pendulum rod, v , and w be the translational and angular velocity of the cart and pendulum. The physical model of the system is shown in fig (1).
Therefore,

$$x_1 = x + l\sin\theta$$
$$y_1 = l\cos\theta$$
$$x_1 = \dot{x} + l\dot{\theta}\cos\theta$$
$$\dot{y}_1 = l\dot{\theta}\sin\theta$$

Let V1 be the resultant velocity of pendulum & cart,

$V_1^2 = \dot{x}_1^2 + \dot{y}_1^2$

$= \dot{x}2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2$

Therefore kinetic energy of the pendulum,

$$k_1 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$= \frac{1}{2}m(\dot{x}^2 + 2lx\theta\cos\theta + ml^2\dot{\theta}^2) + \frac{1}{2}I\dot{\theta}^2$

Kinetic energy of the of the cart,

$$k_2 = \frac{1}{2}M\dot{x}^2$$

Now the Potential energy of the pendulum,

$P_1 = mgy_1 = mgl\cos\theta$

Potential energy of the cart $P_2 = 0$

The Lagrangian of the entire system is given as,

L =kinectic energy-potential energy

L= $\frac{1}{2}$(mẋ2 +2mlẋ$\dot{\theta}$cos□ +ml^2$\dot{\theta}$2+Mẋ2)+ $\frac{1}{2}$I$\dot{\theta}$2]-mglcos□

The Euler- Lagrangian equations are given by,

$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\theta}}\right) - \frac{\delta L}{\delta\theta} + d\dot{\theta} = 0$ and

$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{x}}\right) - \frac{\delta L}{\delta x} + b\dot{x} = F$

Simplifying the above equations we get

$(I + ml^2)\ddot{\theta} + ml\cos\theta\,\ddot{x} - mgl\sin\theta + d\dot{\theta} = 0$ ……..(1)

$(M + m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 + b\dot{x} = F$ ……(2)

The above equation shows the dynamics of the entire system. In order to derive the linear differential equation modelling, we need to linearaize the non linear differtional equation obtained as above so far. For small angle deviation around the upright equilibrium (fig.2) point assume

$\sin\theta = \theta, \cos\theta = 1, \dot{\theta}^2 = 0$

Using above relation we can write as,

$r\ddot{\theta} + q\ddot{x} - k\theta + d\dot{\theta} = 0$……..(3)

$p\ddot{x} + q\ddot{\theta} + b\dot{x} = F$ …….(4)

Where, (M + m)= p, mgl=k, ml=q, I + ml^2= r

Eq (3&4) is the linear differential equation modeling of the entire system.

3. STATE SPACE MODELLING

Let,$\theta = x_1$, $\dot{\theta} = x_2 = \dot{x}_1$, $\ddot{\theta} = \dot{x}_2 = \ddot{x}_1$ and, $x = x_3$, $\dot{x} = x_4 = \dot{x}_3$, $\ddot{x} = \dot{x}_4 = \ddot{x}_3$
From state space modeling, the system matrices are found in matrix from given below.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0.0000000 \\ 4.0088 & -0.047753952 & 0 & 0.0139155 \\ 0 & 0 & 0 & 1.0000000 \\ 0.116806166 & 0.0939155 & 0 & 0.0344200 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 0 \\ -0.27831 \\ 0 \\ 0.68842 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$, Y be the output equation

4. STATE FEEDBACK CONTROLLER DESIGN CONDITIONS:

Our problem is to have a closed loop system having an overshoot of 10% and settling time of 1 sec. Since the overshoot

$MP = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.1$.

Therefore, ξ = 0.591328 and wn=6.7644 rad/sec. The dominant poles are at=-4 ± j5.45531, the third and fourth pole are placed 5 & 10 times deeper into the s-plane than the dominant poles. Hence the desired characteristics equation:-

s4 + 68s3 +845.7604s2 +9625.6s + 36608.32 =0

Let gain, k= [k1 k2 k3 k4]

A – Bk=

$$A - Bk = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -k_1 & 0.2346-k_2 & 6.8963-k_3 & -0.0765-k_4 \end{pmatrix}$$

Closed loop characteristics equation:

S4 +(0.0765-k4)s3 +(-6.8963 + k3)s2 +(-0.2346 +k2)s +k1 = 0

Comparing all the coefficient of above equation we found

K 1 = [36608.32 9625.8346 852.6567 -67.9235]

Similarly if the poles are are placed 2 & 3 times and also 12 & 14 times deeper into the s-plane than the dominant poles then we got the value of k 2 & k3 as,

K 2= [123002.88 26263.2746 2823.8263 -111.9235]

K 3= [4392.96 859.7346 308.6563 -27.9235]

Where, K1, K2, K3 and K4 are gain vectors for different sets of desire poles.

Table 1: Parameters of the system from feedback instrument .U.K.

Parameter	Value	Unit
Cart mass(M)	1.206	Kilo gram
Mass of the pendulum(m)	0.2693	Kilo gram
Half Length of pendulum(l)	0.1623	Meter
Coefficient of frictional force(b)	0.05	Ns/m
Pendulum damping coefficient(q)	0.005	Mm/rad
Moment of inertia of pendulum(I)	0.099	kg/m^2
Gravitation force(g)	9.8	m/s^2

5. LQR DESIGN

A system can be expressed in state variable form as

$\dot{x} = Ax + Bu$

with $x(t) \in R^N$, $u(t) \in R^N$. The initial condition is x(0) . We assume here that all the states are measurable and seek to find a state-variable feedback (SVFB) control

$u = -Kx + v$

that gives desired closed-loop properties.
The closed-loop system using this control becomes

$\dot{x} = (A- BK)\, x+ Bv= Ac\, x+ Bv$

with Ac the closed-loop plant matrix and v the new command input.

Ackermann's formula gives a SVFB K that places the poles of the closed-loop system at desired location . To design a SVFB that is optimal, we may define the performance index J as

$$J = \frac{1}{2}\int_0^\infty X^T (Q + K^T RK)X dt$$

We assume that input v(t) is equal to zero since our only concern here are the internal stability properties of the closed loop system.

The objective in optimal design is to select the SVFB K that minimizes the performance index J.

The two matrices Q (an n× n matrix) and R (an m × n matrix) are of appropriate dimension.

One should select Q to be positive semi-definite and R to be positive definite. Since the plant is linear and the performance index is quadratic, the problem of determining the SVFB K to minimize J is called the Linear Quadratic Regulator (LQR).To find the optimal feedback gain matrix K we proceed as follows. Suppose there exists a constant matrix P such that

$$\frac{d}{dt}(x^T Px) = -x^T (Q + K^T RK)X$$

After some mathematical manipulation, the equation becomes,

$$J = -\frac{1}{2}\int_0^\infty \frac{d}{dt}(x^T Px)dt = \frac{1}{2}x^T(0)Px(0)$$

Where, we assumed that the closed-loop system is stable so that X^T goes to zero as time t goes to infinity. Substituting the values we get,

$$\dot{x^T}Px + x^T P\dot{x} + x^T Qx + x^T K^T RKx = 0$$

$$x^T A^T{}_c Px + x^T PA_c x + x^T Qx + x^T K^T RKx = 0$$

$$x^T (A_c^T P + PA_c + Q + K^T RKx)x = 0$$

It has been assumed that the external control v(t) is equal to zero. Now note that the last equation has to hold for every X^T. Therefore, the term in brackets must be identically equal to zero. Thus, proceeding one sees that

$$(A - BK)^T P + P(A - BK) + Q + K^T RK = 0$$

$$A^T P + PA + Q + K^T RK - K^T B^T P - PBK = 0$$

This is a matrix quadratic equation. Exactly as for the scalar case, one may complete the squares. Though this procedure is a bit complicated for matrices, suppose we select

$$K = R^{-1}B^T P$$

Then, there results

$$A^T P + PA + Q - PBR^{-}B^T P = 0$$

This result is of extreme importance in modern control theory. The above Equation is known as the algebraic Riccati equation (ARE). It is a matrix quadratic equation that can be solved for the auxiliary matrix P given (A,B,Q,R).
The design procedure for finding the LQR feedback K is:

1. Select design parameter matrices Q and R
2. Solve the algebraic Riccati equation for P
3. Find the SVFB using $K = R^{-1}B^T P$

6 SIMULATION & RESULTS

6.1.a Response due to LQR

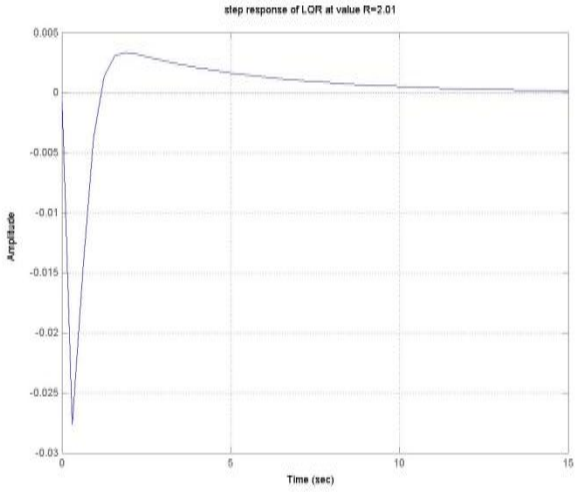


Fig2:- LQR response

Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 10]

R=0.1000

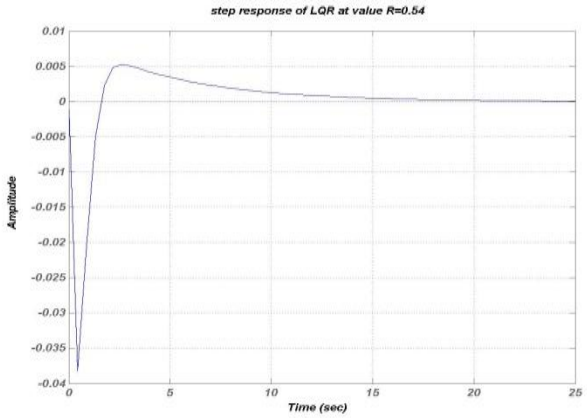


Fig3:- LQR response

Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 10]

R=0.033

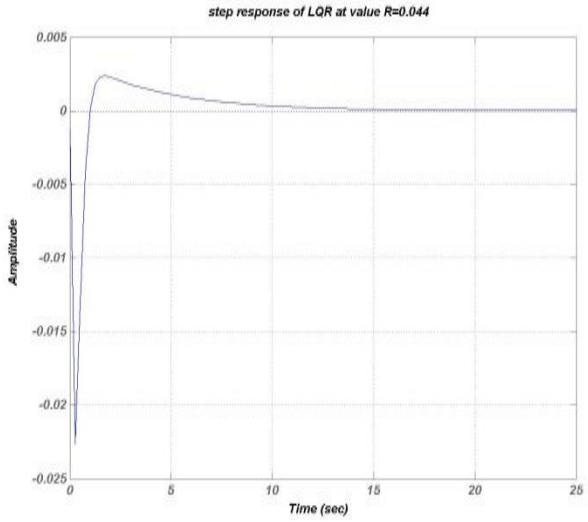


Fig4:- LQR response

Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 10]

R=0.044

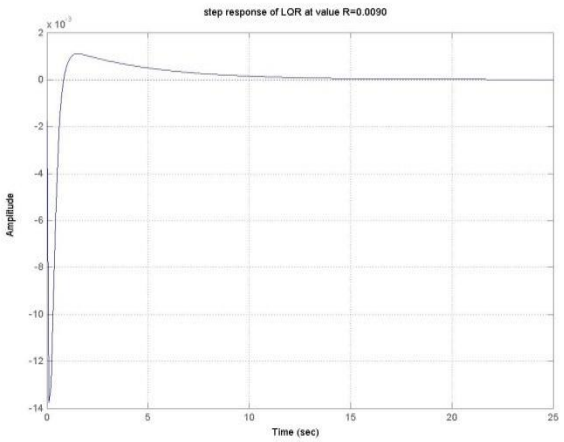


Fig5:- LQR response

Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 10]

R=0.088

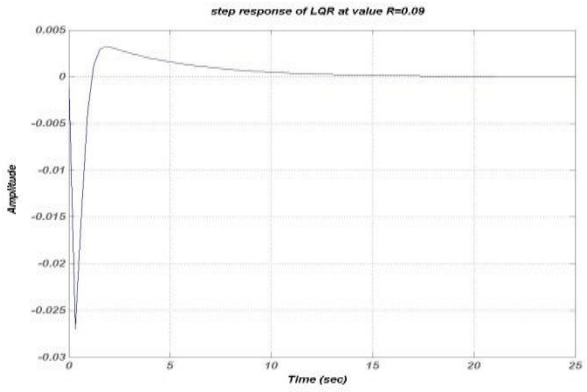
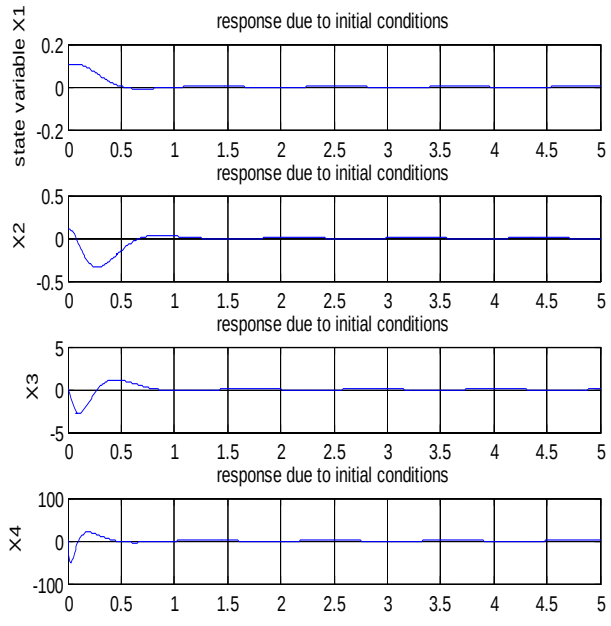


Fig6:- LQR response

Q=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 10]

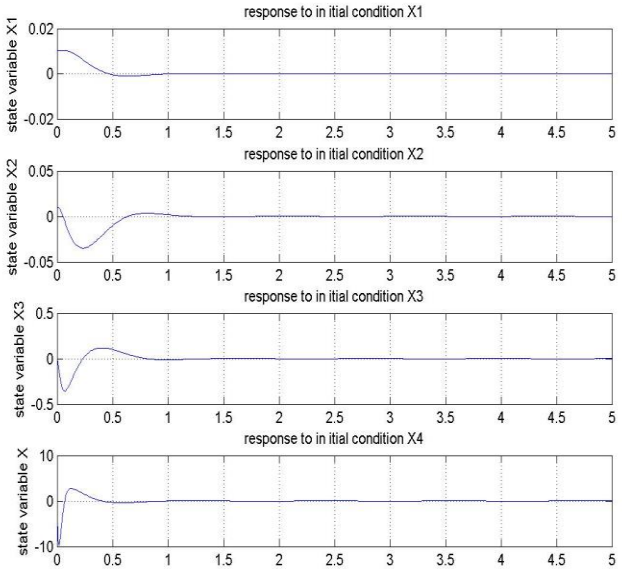
R=0.090

6.2.a-Stabilisation of Angle of the Pendulum by State Feedback Controller with Initial Condition



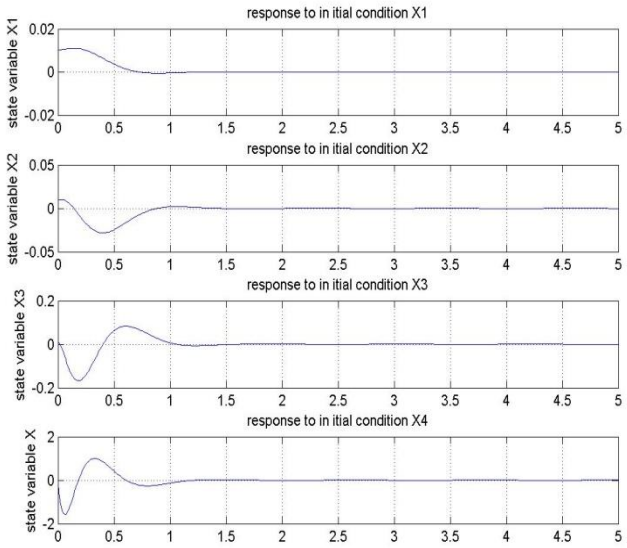
K 1 = [36608.32 9625.8346 852.6567 -67.9235]

Fig7:- Response of state feedback controller considering initial condition when poles are placed 5 & 10 times deeper into the s-plane.



K 2= [123002.88 26263.2746 2823.8263 -111.9235]

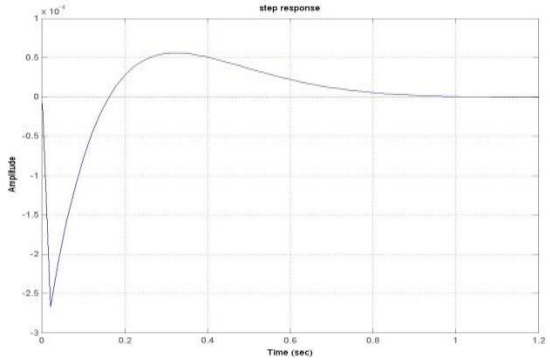
Fig8:- Response of state feedback controller considering initial condition when poles are placed 12 & 14 times deeper into the s-plane.



K 3= [4392.96 859.7346 308.6563 -27.9235]

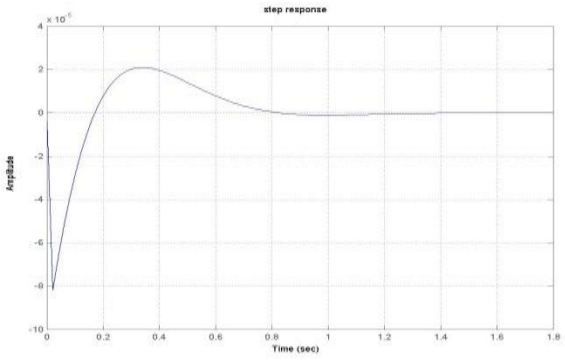
Fig9:- Response of state feedback controller considering initial condition when poles are placed 2 & 3 times deeper into the s-plane.

6.2.b-Step Response of the System by State Feedback Controller



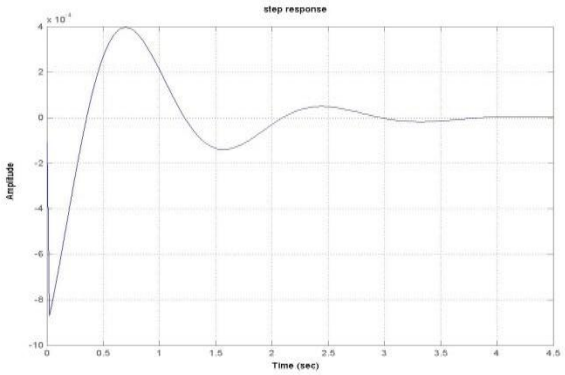
K 1 = [36608.32 9625.8346 852.6567 -67.9235]

Fig10:- Response of Angle as output using state feedback controller



K 2= [123002.88 26263.2746 2823.8263 -111.9235]

Fig11:- Response of Angle as output using state feedback controller



K 3= [4392.96 859.7346 308.6563 -27.923]

Fig12:- Response of Angle as output using state feedback

CONCLUSIONS

Modeling of inverted pendulum shows that system is unstable with non-minimum phase zero.Results of applying state feedback controllers show that the system can be stabilized. while LQR controller method is cumbersome because of selection of constants of controller. Constant of the controllers can be tuned by some soft computing techniques for better result.Fuzzy logic controller can be use in equation (1&2) would help finding out the solution of non-linear differential equations thus helping towards the design of non-linear controller.

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BIOGRAPHIES

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Jagdeo Mahto was born in Madhubani, Bihar, India, in 1943. He obtained the B.Sc (Engg) degree in Electrical Engineering from Bhagalpur University in 1964, M.Tech. in Control System from IIT Kharagpur, India in 1970 and Ph.D in Control System in 1984 from IIT Delhi, India. He served MIT Muzaffarpur from 1964 to 1971 in the capacity of Lecturer and Assistant Professor. From 1971 to 1980 he served as Asst. Professor, from 1980 to 1985 as Associate Professor and from 1985 to 1988 as Professor in the Department of Electrical Engineering at BIT Sindri, India. He taught at Bright Star University, Brega (Libya) from 1988 to 1989. From 1989 to 2003 he was again at BIT Sindri. From 2004 till date he is Professor at Asansol Engineering College, India.

Pankaj Kumar was born in Muzaffarpur, India, in 1970 and received the B.Sc and M.Sc. degree in Electronics Honours and Electronic Science respectively from Magadh University and Gauhati University Assam. He received M.Sc Engineering in Control System Engineering from Patna University in 2004. He began his career as Lecturer in Bihar University Muzaffarpur. Currently he is an Assistant Professor in the Department of Electrical Engineering, Asansol Engineering College, Asansol, India.