SEMICOMPATIBILITY AND FIXED POINT THEOREM IN FUZZY METRIC SPACE USING IMPLICIT FUNCTION

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Abstract

In this paper we proved fixed point theorem of four mapping on fuzzy metric space based on the concept of semi copatibility using implicit relation. These results generalize several corresponding relations in fuzzy metric space. All the results of this paper are new.

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1. INTRODUCTION

Semicompatible maps in d-topological space introduced by Cho et al.[2].They define a pair of self maps to be compatible if conditions (i) (,*ST SyTy*=implies that ; (ii) for sequence {*STyTSy*=}*nx*in *X*and *xX*∈,whenever {}*nSxx*→,{}*nTxx*→,then as ,hold. However, in fuzzy metric space (ii) implies (i), taking *nSTxTx*→*n*→∞*nxy*=for all and*nxTySy*==.So we define a semecompatible pair of self maps in fuzzy metric space by condition (ii) only. Saliga [9] and Sharma et. al [10] proved some interesting fixed point results using implicit real functions and semicompatibility in d-complete topological spaces. Recently, Popa in [8] used the family of implicit real functions to find the fixed points of two pairs of semicompatible maps in a d-complete topological space. Here, denotes the family of all real continuous functions 4*F*4*F*

()4:*FRR*+→ satisfying the following properties.

(i) There exists such that for every with or we have. 1*h*≥0,0*uv*≥≥(),,,0*Fuvuv*≥(),,,0,*Fuvvu*≥*uhv*≥

(ii) , for all . (),,0,00*Fuu*<0*u*>

Jungck and Rhoades [6], Dhage [3] termed a pair of self maps to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. This concept is most general among all the commutativity concepts in this field as every pair of commuting self maps is R-weakly commuting, each pair of R-weakly commuting self maps is compatible and each pair of compatible self maps is weak compatible but reverse is not always true. Similarly, every semicompatible pair of self maps is weak compatible but the reverse is not always true. The main object of this paper is to obtain some fixed point theorems in the setting of fuzzy metric space using weak compatibility, semicompatibility, and an implicit relation.

2. PRELIMINARIES

Definition 2.1. A binary operation is called a continuous *t*-norm if 2*:[0,1][0,1]→

([0,1],∗) is an abelian topological monoid with unit 1 such that $a*b \le c*d$ whenever

$a \le c$ and $b \le d$ for all a,b, c, and [0,1]*d*∈.

Examples of *t*-norm are $a*b = ab$ and $a*b = \min\{a,b\}$.

Definition 2.2 (Kramosil andMich´alek [7]). The 3-tuple $(X,M,*)$ is called a fuzzy metric

space if X is an arbitrary set, ∗ is a continuous *t*-norm, and$M$ is a fuzzy set in 2[0,)*X*×∞

satisfying the following conditions for all ,,*xyzX*∈ and $s,t > 0$:

(FM-1) $M(x, y,0) = 0$;

(FM-2) $M(x, y, t) = 1$, for all $t > 0$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y,x, t)$;

(FM-4) $M(x, y, t)*M(y,z, s) \ge M(x,z, t+s)$;

(FM-5) $M(x, y,\cdot) : [0,\infty)\to[0,1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with

respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

Example 2.3 (George and Veeramani [4]). Let (*X*,*d*) be a metric space. Define *a*∗*b* =

min*{a,b}* and for all ,,*abX*∈

()()(),,,0,,,00.,*tMxyttMxytdxy*=∀>+

Then (*X*,*M*,∗) is a fuzzy metric space. It is called the fuzzy metric space induced by the metric *d*.

Lemma 2.4 (Grabiec [5]). *For all* ,*xyX*∈*, M*(*x, y*,·) *is a nondecreasing function.*

Definition 2.5 (Grabiec [5]). Let (*X*,*M*,∗) be a fuzzy metric space. A sequence *{xn}* in

X is said to be convergent to a point *x* _ *X* if lim*n*→∞*M*(*xn*,*x*, *t*) = 1 for all *t* > 0. Further,

the sequence {}*nx* is said to be a Cauchy sequence in *X* if ()lim,,1*nnnpMxxt*→∞+= for all

t > 0 and *p* > 0. The space is said to be complete if every Cauchy sequence in it convergesto a point of it.

Remark 2.6. Since ∗ is continuous, it follows from (FM-4) that the limit of a sequence in a fuzzy metric space is unique.

In this paper, (*X*,*M*,∗) is considered to be the fuzzy metric space with condition

(FM-6) lim*t*→∞*M*(*x*, *y*, *t*) = 1, for all ,*xyX*∈

Lemma 2.7 (Cho [1]). *Let {yn} be a sequence in a fuzzy metric space* (*X*,*M*,∗) *with the condition (FM-6). If there exists a numbe* ()0,1*k*∈ *such that M*(*yn*+2, *yn*+1,*kt*) ≥*M*(*yn*+1, *yn*, *t*), *for all t* > 0, *then {yn} is a Cauchy sequence in X.*

Lemma 2.8. *Let A and B be two self-maps on a complete fuzzy metric space* (*X*,*M*,∗) *such that for some* , *for all* (0,1*k*∈ ,*xyX*∈ *and t* > 0, *M*(*Ax*,*By*,*kt*) ≥ Min_*M*(*x*, *y*, *t*),*M*(*Ax*,*x*, *t*)_. (2.2)

Then A and B have a unique common fixed point in X.

Proof. Let . Taking *pX*∈0*xp*=, define sequence {}*nx* in *X* by 22*nnAxx* +=and 2122*nnBxx*+= . By taking 22,*nnxxyx*+==and 22,*n xxyx*−==, respectively, in the contractive condition, we obtain that ()(11,,,,*nnnn MxxktMxxt*+−≥, 0,*t*∀>∀ (2.3)

Therefore by Lemma 2.7, {}*nx*is a Cauchy sequence in *X*, which is complete. Hence, {}*nx*converges to some *u* in *X*. Taking 2*nxx*= and *y* = *u* and letting *n*→∞ in the contractive condition, we get *Bu* = *u*. Similarly, by putting *x* = *u* and 21*nyx*+=, we get *Au* = *u*. Therefore, *u* is the common fixed point of the maps *A* and *B*. The uniqueness of the common fixed point follows from the contractive condition. _

Definition 2.9. Let *A* and *S* be mappings from a fuzzy metric space (*X*,*M*,∗) into itself.

Themappings are said to be weak compatible if they commute at their coincidence points, that is, *Ax* = *Sx* implies that *ASx* = *SAx*.

Definition 2.10. Let *A* and *S* be mappings from a fuzzy metric space (*X*,*M*,∗) into itself.

Then the mappings are said to be compatible if (lim,,1,*nnnMASxSAxt*→∞= 0*t*∀> (2.4) whenever {}*nx* is a sequence in *X* such that limlim.*nnnnAxSxxX*→∞→∞== (2.5)

Proposition 2.11 [12]. Self-mappings A and S of a fuzzy metric space (X,M,∗) are compatible, then they are weak compatible. The converse is not true.

Definition 2.12. Let A and S be mappings from a fuzzy metric space (X,M,∗) into itself. Then the mappings are said to be semicompatible if ()lim,,1,*nnMASxSxt*→∞= 0*t*∀> (2.6) whenever {}*nx* is a sequence in X such that limlim.*nnnnAxSxxX*→∞→∞== (2.7)

2.1. A Class of Implicit Relation.

Let +*R*be the set of all non-negative real numbers. Let Φ be the collection of all function satisfying: ()++→*RR*3:φ

(a) φ is lower semcontinuous in each coordinate variable.

(b) Let ()*vvuu*,,φ≥ or (*vuvu*,, φ≥.Then , for ,where *kvu*≥+∈*Rvu*,()11,1,1>=*k*φ.

(c) ()*uuuu*>,,φ, for . {}0−∈+*Ru*

3. MAIN RESULTS

3.1. Theorem: Let *A*,*B*,*S* and *T* be self mappings of a complete Fuzzy metric space satisfying that
(∗,,*MX*()()*XTXA*⊆, (1)
()()*XSXB*⊆ the pair (is semicompatible and *SA*,()*TB*, is weak compatible; (2)
one of *A* or is continuous. (3)
S
for some Φ∈φthere exists such that for all (1,0∈*k Xyx*∈,and 0>*t* ()()()(){}*ktTyByMtSxAxMtTySxMktbyAxM*,,,,,,,,min,,≥ (4)

Then *A*,*B*,*S* and *T* have a unique common fixed point in *X*.

Proof: Let be any arbitrary point as *Xx*∈0()()*XTXA*⊆ and ()(*XSXB*⊆ there exists *Xxx*∈21, such that 2110,*SxBxTxAx*==.Inductively construct sequences { and }*ny*{}*nx* in *X* such that

22122212212,+++++====*nnnnnnSxBxyTxAxy* for ,...2,1,0=*n*Now using (4) with we get 122,+==*nnxyxx*
()()()(){}*ktTxBxMtSxAxMtTxSxMktBxAxMnnnnnnnn*,,,,,,,min,, 121222122122++++≥
That is
()()()(){}*ktyyMtyyMtyyMktyyMnnnnnnnn*,,,,,,,min,,221221212 22212++++++≥
()(*tyyMktyyMnnnn*,,,,2122212+++≥
Similarly, 322232221222,++++++====*nnnnnnSxBxyTxAxy* using 2212,++==*nnxyxx*
()(*tyyMktyyMnnnn*,,,,22123222++++≥
Thus for any *n* and *t* we have
()(*tyyMktyyMnnnn*,,,,11−+≥
Hence by Lemma [2.7] ,{ is a Cauchy sequence in *X* which is complete.Therefore converges to , its subsequences }*ny*{}*nyXu*∈{}{}{}{}122122,,,++*nnnnTxSxBxAx* also converges to that is *u*
{}*uAxn*→2 , { }*uBxn*→+12
{}*uSxn*→2 , { (5) }*uTxn*→+12
Case I : *S* is continuous. In this case , we have
SuSAxn→2 , (6) *SuxSn*→22
The semicompatibility of the pair ()*SA*, gives
SuASxnn=∞→2lim (7) Step 1: By putting in (4), we obtain 122,+==*nnxySxx*
()()()(){}221221222121,,min,,,,,,,*nnnnnnnnMASxBxktMSSxTx tMASxSSxtMBxTxkt*+++≥
Letting using (5),(6) and (7) and the continuity of the norm * , we have *n*→∞
()()()(){},,min,,,,,,,*MSuuktMSuutMSuSutMuukt*≥
that is
()(){},,min,,,1,1*MSuuktMSuut*≥
(),,1*MSuukt*≥
It is non decreasing
(),,1,0*MSuuktt*≥>
Therefore
.*Suu*= (8)

Step 2: By putting 21,*nxuyx*+== in (4)
We obtain that
()()()(){}21212121,,min,,,,,,,*nnn MAuBxktMSuTxtMAuSutMBxTxkt*+++≥
Taking limit as and using (5) and (8) *n*→∞
()(){},,min1,,,,1*MAuuktMAuut*≥
(),,1*MAuut*≥ for 0*t*>
Therefore
Auu=
Hence . (9) *AuuSu*== Step 3:

As ()()*AXTX*⊆ there exists *wX*∈such that *AuSuuTw*===.
By putting 2,*nxxyw*== in (4)
We obtain
()()()(){}2222,,min,,,,,,,*nnnnMAxBwktMSxTwtMAxSxtMBwT wkt*≥
Taking limit as and using (5) we get *n*→∞
()(){},,min1,1,,,*MuBwktMBwukt*≥
We have for all (),,*MuBukt*≥ 0*t*>
Hence (),,*MuBut*=
Thus *u Bw*=
Therefore *BwTwu*==
Since is weak compatible. (,*BT*
We get *TBwBTw*=
that is (10) .*BuTu*=

Step 4:
By putting ,*xuyu*==in (4),(9) and (10)
We obtain that
()()()(){},,min,,,,,,,*MAuBuktMSuTutMAuSutMBuTukt*≥
that is ()(){},,min,,,1,1*MAuBuktMAuBut*≥
(),,*MAuBut*≥ for all 0*t*>
Thus (),,*MAuBut*=
We have .*AuBu*=
Therefore *uA*. *uSuBuTu*====
That is *u* is a common fixed point of ,,,.*ABST* Case II : *A* is continuous. We have the semicompatibility of the pair 2*nASxAu*→(),*AS* gives. 2*nASxSu*→
By uniqueness of limit in fuzzy metric space we obtain that .*AuSu*=

Step 5:
By putting 21,*nxuyx*+== in (4) we obtain that
()()()(){}21212121,,,,,,,,,*nnn MAuBxktminMSuTxtMAuSutMBxTxkt*+++≥
Taking limit as and using (5) and (8) we get *n*→∞
()(){},,min1,,,,1*MAuuktMAuut*≥
We have for all (),,1*MAuut*≥0*t*>
Which gives .*uAu*=
Uniqueness:
Let be another common fixed point of *z*,,,*ABST*.
Then *zAzBzSzTz*==== putting *xu*= and *yz*= in (4) we get
()()()(){},,min,,,,,,,*MAuBzktMSuTztMAuSutMBzTzkt*≥
that is
()(){},,,,,1,1*MuzktMuzt*≥
Therefore we have (),,1*Muzt*≥ for all 0*t*>
Hence (),,1*Muzt*=
That is*uz*. =
Therefore *u* is the unique common fixed point of the self maps ,,,.*ABST*

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