A SEMI-CIRCLE THEOREM IN MAGNETO-ROTATORY THERMOSOLUTAL CONVECTION IN RIVLIN-ERICKSEN VISCOELASTIC FLUID IN A POROUS MEDIUM

Ajaib S. Banyal

Department of Mathematics, Govt. College Nadaun (Hamirpur), (HP) INDIA 177033
***ajaibbanyal@rediffmail.com**, Fax No. : 01972232688*

Abstract

Thermosolutal convection in a layer of Rivlin-Ericksen viscoelastic fluid of Veronis (1965) type is considered in the presence of uniform vertical magnetic field and rotation in a porous medium. Following the linearized stability theory and normal mode analysis, the paper through mathematical analysis of the governing equations of Rivlin-Ericksen viscoelastic fluid convection in the presence of uniform vertical magnetic field and rotation, for any combination of perfectly conducting free and rigid boundaries of infinite horizontal extension at the top and bottom of the fluid, established that the complex growth rate σ of oscillatory perturbations, neutral or unstable for all wave numbers, must lie inside right half of the a semi-circle

$$\sigma_r^{\ 2} + \sigma_i^{\ 2} \langle Maximumof \left[\left\{ \frac{1}{4}\left(\frac{\varepsilon P_l}{P_l + \varepsilon F}\right)^2 \left(Q + \sqrt{Q^2 + 4T_A}\right)^2 \right\}, \left\{ \frac{R_s}{E^{'} p_3}\left(\frac{\varepsilon P_l}{P_l + \varepsilon F}\right) \right\} \right],$$

in the $\sigma_r \sigma_i$-plane, where R_s is the thermosolutal Rayleigh number, T_A is the Taylor number, F is the viscoelasticity parameter, p_3 is the thermosolutal prandtl number, ε is the porosity and P_l is the medium permeability. This prescribes the bounds to the complex growth rate of arbitrary oscillatory motions of growing amplitude in the Rivlin-Ericksen viscoelastic fluid in Veronis (1965) type configuration in the presence of uniform vertical magnetic field and rotation in a porous medium. A similar result is also proved for Stern (1960) type of configuration. The result is important since the result hold for any arbitrary combinations of dynamically free and rigid boundaries.

Keywords: *Thermosolutal convection; Rivlin-Ericksen Fluid; Magnetic field; Rotation; PES; Rayleigh number; Chandrasekhar number; Taylor number.*

MSC 2000 No.: 76A05, 76E06, 76E15; 76E07; 76U05.

---

1. INTRODUCTION

Chandrasekhar [1] in his celebrated monograph presented a comprehensive account of the theoretical and experimental study of the onset of Bénard Convection in Newtonian fluids, under varying assumptions in hydrodynamics and hydromagnetics. The use of Boussinesq approximation has been made throughout, which states that the density changes are disregarded in all other terms in the equation of motion except the external force term. The problem of thermohaline convection in a layer of fluid heated from below (above) and subjected to a stable (destabilizing) salinity gradient has been considered by Veronis [2] and Stren [3] respectively. The physics is quite similar in the stellar case, in that helium acts like in raising the density and in diffusing more slowly than heat. The problem is of great importance because of its applications to atmospheric physics and astrophysics, especially in the case of the ionosphere and the outer layer of the atmosphere. The thermosolutal convection problems also arise in oceanography, limnology and engineering. Bhatia and Steiner [4] have considered the effect of uniform rotation on the thermal instability of a viscoelastic (Maxwell) fluid and found that rotation has a destabilizing influence in contrast to the stabilizing effect on Newtonian fluid. Sharma [5] has studied the thermal instability of a layer of viscoelastic (Oldroydian) fluid acted upon by a uniform rotation and found that rotation has destabilizing as well as stabilizing effects under certain conditions in contrast to that of a Maxwell fluid where it has a destabilizing effect. There are many elastico-viscous fluids that cannot be characterized by Maxwell's constitutive relations or Oldroyd's [6] constitutive relations. Two such classes of fluids are Rivlin-Ericksen's and Walter's (model B') fluids. Rivlin-Ericksen [7] has proposed a theoretical model for such one class of elastico-viscous fluids. Sharma and kumar [8] have studied the effect of rotation on thermal instability in Rivlin-Ericksen elastico-viscous fluid and found that rotation has a stabilizing effect and introduces oscillatory modes in the system. Kumar et al. [9] considered effect of rotation and magnetic field on Rivlin-Ericksen elastico-viscous fluid and found that rotation has stabilizing effect; where as magnetic field has both stabilizing and destabilizing effects. A layer of such fluid heated from below or under the action of magnetic field or rotation or both may find applications in geophysics, interior of the Earth, Oceanography, and the atmospheric physics. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable.

In all above studies, the medium has been considered to be non-porous with free boundaries only, in general. In recent years, the investigation of flow of fluids through porous media has become an important topic due to the recovery of crude oil from the pores of reservoir rocks. When a fluid permeates a porous material, the gross effect is represented by the Darcy's law. As a result of this macroscopic law, the usual viscous term in the equation of Rivlin-Ericksen fluid motion is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)q\right]$, where μ and μ' are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid, k_1 is the medium permeability and q is the Darcian (filter) velocity of the fluid. The problem of thermosolutal convection in fluids in a porous medium is of great importance in geophysics, soil sciences, ground water hydrology and astrophysics. Generally, it is accepted that comets consist of a dusty 'snowball' of a mixture of frozen gases which, in the process of their journey, changes from solid to gas and vice-versa. The physical properties of the comets, meteorites and interplanetary dust strongly suggest the importance of non-Newtonian fluids in chemical technology, industry and geophysical fluid dynamics. Thermal convection in porous medium is also of interest in geophysical system, electrochemistry and metallurgy. A comprehensive review of the literature concerning thermal convection in a fluid-saturated porous medium may be found in the book by Nield and Bejan [10].

Pellow and Southwell [11] proved the validity of PES for the classical Rayleigh-Bénard convection problem. Banerjee et al [12] gave a new scheme for combining the governing equations of thermohaline convection, which is shown to lead to the bounds for the complex growth rate of the arbitrary oscillatory perturbations, neutral or unstable for all combinations of dynamically rigid or free boundaries and, Banerjee and Banerjee [13] established a criterion on characterization of non-oscillatory motions in hydrodynamics which was further extended by Gupta et al. [14]. However no such result existed for non-Newtonian fluid configurations in general and in particular, for Rivlin-Ericksen viscoelastic fluid configurations. Banyal [15] have characterized the oscillatory motions in Rivlin-Ericksen fluid in the presence of rotation.

Keeping in mind the importance of non-Newtonian fluids, as stated above, the present paper is an attempt to prescribe the bounds to the complex growth rate of arbitrary oscillatory motions of growing amplitude, in a thermosolutal convection of a layer of incompressible Rivlin-Ericksen fluid configuration of Veronis [2] type in the presence of uniform vertical magnetic field and rotation in a porous medium, when the bounding surfaces are of infinite horizontal extension, at the top and bottom of the fluid and are with any arbitrary combination of perfectly conducting dynamically free and rigid boundaries. A similar result is also proved for Stern [3] type of configuration. The result is important since the result hold for any arbitrary combinations of dynamically free and rigid boundaries.

2. FORMULATION OF THE PROBLEM AND PERTURBATION EQUATIONS

Here we Consider an infinite, horizontal, incompressible electrically conducting Rivlin-Ericksen viscoelastic fluid layer, of thickness d, heated from below so that, the temperature, density and solute concentrations at the bottom surface z = 0 are T_0, ρ_0 and C_0 and at the upper surface z = d are T_d, ρ_d and C_d respectively, and that a uniform adverse temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ and a uniform solute gradient $\beta'\left(=\left|\frac{dC}{dz}\right|\right)$ is maintained. The gravity field $\vec{g}(0,0,-g)$, uniform vertical rotation $\vec{\Omega}(0,0,\Omega)$ and a uniform vertical magnetic field pervade on the system $\vec{H}(0,0,H)$. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1.

Let p, ρ, T, C, α, α', g, η, μ_e and $\vec{q}(u,v,w)$ denote respectively the fluid pressure, fluid density temperature, solute concentration, thermal coefficient of expansion, an analogous solvent coefficientof expansion, gravitational acceleration, resistivity, magnetic permeability and filter velocity of the fluid. Then the momentum balance, mass balance, and energy balance equation of Rivlin-Ericksen fluid and Maxwell's equations through porous medium, governing the flow of Rivlin-Ericksen fluid in the presence of uniform vertical magnetic field and uniform vertical rotation (Rivlin and Ericksen [7]; Chandrasekhar [1] and Sharma et al [16]) are given by

$$\frac{1}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}\left(\vec{q}.\nabla\right)\vec{q}\right]=-\left(\frac{1}{\rho_0}\right)\nabla p+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)-\frac{1}{k_1}\left(\nu+\nu'\frac{\partial}{\partial t}\right)\vec{q}$$

$$+\frac{\mu_e}{4\pi\rho_o}(\nabla\times\vec{H})\times\vec{H}+\frac{2}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right), \quad (1)$$

$$\nabla.\vec{q}=0, \quad (2)$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=\kappa\nabla^2 T, \quad (3)$$

$$E'\frac{\partial C}{\partial t}+(\vec{q}.\nabla)C=\kappa'\nabla^2 C \quad (4)$$

$$\varepsilon\frac{d\vec{H}}{dt}=(\vec{H}.\nabla)\vec{q}+\varepsilon\eta\nabla^2\vec{H}, \quad (5)$$

$$\nabla.\vec{H}=0, \quad (6)$$

Where $\frac{d}{dt}=\frac{\partial}{\partial t}+\varepsilon^{-1}\vec{q}.\nabla$, stands for the convective derivatives. Here

$E=\varepsilon+(1-\varepsilon)\left(\frac{\rho_s c_s}{\rho_0 c_i}\right)$, is a constant and E' is a constant analogous to E but corresponding to solute rather than heat, while ρ_s, c_s and ρ_0, c_i, stands for the density and heat capacity of the solid (porous matrix) material and the fluid, respectively, ε is the medium porosity and $\vec{r}(x,y,z)$.

The equation of state is

$$\rho=\rho_0\left[1-\alpha(T-T_0)+\alpha'(C-C_0)\right], \quad (7)\backslash$$

Where the suffix zero refer to the values at the reference level z = 0. In writing the equation (1), we made use of the Boussinesq approximation, which states that the density variations are ignored in all terms in the equation of motion except the external force term. The kinematic viscosity ν, kinematic viscoelasticity ν', magnetic permeability μ_e, thermal diffusivity κ, the solute diffusivity κ', and electrical resistivity η, and the coefficient of thermal expansion α are all assumed to be constants.

The steady state solution is

$$\vec{q}=(0,0,0), \quad \rho=\rho_0(1+\alpha\beta z-\alpha'\beta' z),$$
$$T=-\beta z+T_0,\ C=-\beta' z+C_0 \quad (8)$$

Here we use the linearized stability theory and the normal mode analysis method. Consider a small perturbations on the steady state solution, and let $\delta\rho$, δp, θ, γ, $\vec{q}(u,v,w)$ and $\vec{h}=(h_x,h_y,h_z)$ denote respectively the perturbations in density ρ, pressure p, temperature T, solute concentration C, velocity $\vec{q}(0,0,0)$ and the magnetic field $\vec{H}=(0,0,H)$. The change in density $\delta\rho$, caused mainly by the perturbation θ and γ in temperature and concentration, is given by

$$\delta\rho=-\rho_0(\alpha\theta-\alpha'\gamma). \quad (9)$$

Then the linearized perturbation equations of the Rinlin-Ericksen fluid reduces to

$$\frac{1}{\varepsilon}\frac{\partial\vec{q}}{\partial t}=-\frac{1}{\rho_0}(\nabla\delta p)-\vec{g}(\alpha\theta-\alpha'\gamma)-\frac{1}{k_1}\left(\nu+\nu'\frac{\partial}{\partial t}\right)\vec{q}+\frac{\mu_e}{4\pi\rho_0}\left(\nabla\times\vec{h}\right)\times\vec{H}+\frac{2}{\varepsilon}\left(\vec{q}\times\vec{\Omega}\right) \quad (10)$$

$$\nabla.\vec{q}=0, \quad (11)$$

$$E\frac{\partial\theta}{\partial t}=\beta w+\kappa\nabla^2\theta\,, \tag{12}$$

$$E^{'}\frac{\partial\gamma}{\partial t}=\beta^{'}w+\kappa^{'}\nabla^2\gamma\,, \tag{13}$$

$$\varepsilon\frac{\partial\vec{h}}{\partial t}=\left(\vec{H}.\nabla\right)\vec{q}+\varepsilon\eta\nabla^2\vec{h}\,. \tag{14}$$

And $\nabla.\vec{h}=0$, (15)

Where $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

3. NORMAL MODE ANALYSIS

Analyzing the disturbances into two-dimensional waves, and considering disturbances characterized by a particular wave number, we assume that the Perturbation quantities are of the form

$$[w,\theta,h_{z,}\gamma,\varsigma,\xi]=[W(z),\Theta(z),K(z),\Gamma(z),Z(z),X(z)]\exp\left(ik_x x+ik_y y+nt\right), \tag{16}$$

Where k_x, k_y are the wave numbers along the x- and y-directions, respectively, $k=\left(k_x^{\,2}+k_y^{\,2}\right)^{\frac{1}{2}}$, is the resultant wave number, n is the growth rate which is, in general, a complex constant and $\varsigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ and $\xi=\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}$ denote the z-component of vorticity and current density respectively and $W(z), K(z), \Theta(z), \Gamma(z), Z(z)$ and $X(z)$ are the functions of z only.

Using (16), equations (10)-(15), within the framework of Boussinesq approximations, in the non-dimensional form transform to

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\left(D^2-a^2\right)W=-Ra^2\Theta+R_s a^2\Gamma-T_A DZ+Q\left(D^2-a^2\right)DK \tag{17}$$

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]Z=DW+QDX\,, \tag{18}$$

$$\left(D^2-a^2-p_2\sigma\right)K=-DW\,, \tag{19}$$

$$\left(D^2-a^2-p_2\sigma\right)X=-DZ\,, \tag{20}$$

$$\left(D^2-a^2-Ep_1\sigma\right)\Theta=-W\,, \tag{21}$$

and

$$\left(D^2-a^2-E^{'}p_3\sigma\right)\Gamma=-W\,, \tag{22}$$

Where we have introduced new coordinates (x', y', z') = (x/d, y/d, z/d) in new units of length d and $D=d/dz'$. For convenience, the dashes are dropped hereafter. Also we have substituted $a=kd, \sigma=\frac{nd^2}{\nu}$, $p_1=\frac{\nu}{\kappa}$ is the thermal Prandtl number; $p_3=\frac{\nu}{\kappa^{'}}$ is the thermosolutal Prandtl number; $p_2=\frac{\nu}{\eta}$ is the magnetic Prandtl number; $P_l=\frac{k_1}{d^2}$ is the dimensionless medium permeability, $F=\frac{\nu^{'}}{d^2}$ is the dimensionless viscoelasticity parameter of the Rivlin-Ericksen fluid; $R=\frac{g\alpha\beta d^4}{\kappa\nu}$ is the thermal Rayleigh number; $R_s=\frac{g\alpha^{'}\beta^{'}d^4}{\kappa^{'}\nu^{'}}$ is the thermosolutal Rayleigh number; $Q=\frac{\mu_e H^2 d^2}{4\pi\rho_0\nu\eta\varepsilon}$ is the Chandrasekhar number and $T_A=\frac{4\Omega^2 d^4}{\nu^2\varepsilon^2}$ is the Taylor number. Also we have Substituted $W=W_{\oplus}, \Theta=\frac{\beta d^2}{\kappa}\Theta_{\oplus}$,

$$\Gamma=\frac{\beta^{'}d^2}{\kappa}\Gamma_{\oplus}, Z=\frac{2\Omega d}{\nu\varepsilon}Z_{\oplus}, K=\frac{Hd}{\varepsilon\eta}K_{\oplus},$$

$X = \left(\frac{Hd}{\varepsilon\eta}\right)\left(\frac{2\Omega d}{\varepsilon\nu}\right)X_{\oplus}$ and $D_{\oplus} = dD$, dropped $(\oplus)$ for convenience.

We now consider the cases where the boundaries are rigid-rigid or rigid-free or free-rigid or free-free at z = 0 and z = 1 respectively, as the case may be, are perfectly conducting and maintained at constant temperature and solute concentration. Then the perturbations in the temperature and solute concentration are zero at the boundaries. The appropriate boundary conditions with respect to which equations (13)--(16), must possess a solution are

W = 0 =K= $\Theta = \Gamma$, on both the horizontal boundaries,
DW = 0=Z=DX, on a rigid boundary,
$D^2W = 0 = DZ$ =X, on a dynamically free boundary, (23)

Equations (17)--(22), along with boundary conditions (23), pose an eigenvalue problem for σ and we wish to characterize σ_i, when $\sigma_r \geq 0$.

4. MATHEMATICAL ANALYSIS

We prove the following lemma:
Lemma 1: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 |\Gamma|^2 dz \leq \frac{1}{E'^2 p_3^2 |\sigma|^2} \int_0^1 |W|^2 dz$$

Proof: Further, multiplying equation (22) and its complex conjugate, and integrating by parts each term on right hand side of the resulting equation for an appropriate number of times and making use of boundary conditions on Γ namely $\Gamma(0) = 0 = \Gamma(1)$ along with (22), we get

$$\int_0^1 \left|\left(D^2 - a^2\right)\Gamma\right|^2 dz + 2E' p_3 \sigma_r \int_0^1 \left(|D\Gamma|^2 + a^2|\Gamma|^2\right) dz + E'^2 p_3^2 |\sigma|^2 \int_0^1 |\Gamma|^2 dz = \int_0^1 |W|^2 dz \qquad (24)$$

Since $\sigma_r \geq 0$ therefore the equation (24) gives,

$$\int_0^1 |\Gamma|^2 dz \leq \frac{1}{E'^2 p_3^2 |\sigma|^2} \int_0^1 |W|^2 dz \qquad (25)$$

This completes the proof of lemma**.**

Lemma 2: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 |\Theta|^2 dz \leq \frac{1}{E^2 p_1^2 |\sigma|^2} \int_0^1 |W|^2 dz$$

Proof: Further, multiplying equation (21) and its complex conjugate, and integrating by parts each term on right hand side of the resulting equation for an appropriate number of times and making use of boundary conditions on Θ namely $\Theta(0) = 0 = \Theta(1)$ along with (21), we get

$$\int_0^1 \left|\left(D^2 - a^2\right)\Theta\right|^2 dz + 2E' p_3 \sigma_r \int_0^1 \left(|D\Theta|^2 + a^2|\Theta|^2\right) dz + E'^2 p_3^2 |\sigma|^2 \int_0^1 |\Theta|^2 dz = \int_0^1 |W|^2 dz \qquad (26)$$

Since $\sigma_r \geq 0$ therefore the equation (26) gives,

$$\int_0^1 |\Theta|^2 dz \leq \frac{1}{E^2 p_1^2 |\sigma|^2} \int_0^1 |W|^2 dz \qquad (27)$$

This completes the proof of lemma.

Lemma 3: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 |Z|^2 dz \langle \frac{1}{|\sigma|^2 \left(\frac{1}{\varepsilon} + \frac{F}{P_l}\right)^2} \int_0^1 |DW|^2 dz$$

Proof: Further, multiplying equation (20) with its complex conjugate, and integrating by parts each term on both sides of the resulting equation for an appropriate number of times and making use of appropriate boundary conditions (23), we get

$$\left[|\sigma|^2 \left(\frac{1}{\varepsilon} + \frac{F}{P_l}\right)^2 + \frac{1}{P_l^2} + \frac{2\sigma_r}{P_l}\left(\frac{1}{\varepsilon} + \frac{F}{P_l}\right)\right] \int_0^1 |Z|^2 dz$$

$$=\int_0^1 |DW|^2 dz + Q^2\int_0^1 |DX|^2 dz + 2Q\,\mathrm{Re}.Partof\left\{\int_0^1 DWDX^* dz\right\} \quad (28)$$

Multiplying the complex conjugate of equation (16) by DX and integrating over the range of z, we get

$$\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^* DXdz = \int_0^1 DXDW^* dz + Q\int_0^1 |DX|^2 dz \quad , \quad (29)$$

Utilizing the equation (18) and appropriate boundary condition DX (0) = 0 = DX (1), we get

$$\int_0^1 DXZ^* dz = -\int_0^1 XDZ^* dz = \int_0^1 X\left(D^2-a^2-p_2\sigma^*\right)X^* dz$$

$$= -\int_0^1 \left(|DX|^2 + a^2|X|^2\right)dz - p_2\sigma^*\int_0^1 |X|^2 dz \quad (30)$$

Substituting (30) in (29), we get

$$\mathrm{Re}.Partof\left\{\int_0^1 DXDW^* dz\right\} = -Q\int_0^1 |DX|^2 dz$$
$$-\left[\frac{\sigma_r}{\varepsilon}+\frac{1}{P_l}(1+\sigma_r F)\right]\left\{\int_0^1 \left(|DX|^2+a^2|X|^2\right)dz + p_2\sigma_r\int_0^1 |X|^2 dz\right\} , \quad (31)$$

Substituting (31) in (28), we get

$$\left[|\sigma|^2\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)^2+\frac{1}{P_l^2}+\frac{2\sigma_r}{P_l}\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)\right]\int_0^1 |Z|^2 dz + Q^2\int_0^1 |DX|^2 dz$$
$$+2Q\left[\frac{\sigma_r}{\varepsilon}+\frac{1}{P_l}(1+\sigma_r F)\right]$$
$$\left\{\int_0^1 \left(|DX|^2+a^2|X|^2\right)dz + p_2\sigma_r\int_0^1 |X|^2 dz\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32)$$
$$=\int_0^1 |DW|^2 dz$$

Now F $\rangle$ 0, $Q\rangle 0$, $p_2\rangle 0$ and $\sigma_r \ge 0$, therefore the equation (32), gives

$$\int_0^1 |Z|^2 dz \langle \frac{1}{|\sigma|^2\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)^2}\int_0^1 |DW|^2 dz \quad , \quad (33)$$

This completes the proof of lemma.

Lemma 4: For any arbitrary oscillatory perturbation, neutral or unstable

$$\int_0^1 \left(|DK|^2 + a^2|K|^2\right)dz \le \frac{1}{p_2|\sigma|}\int_0^1 |DW|^2 dz$$

Proof: Multiplying equation (19) and its complex conjugate, and integrating by parts each term on both sides of the resulting equation for an appropriate number of times and making use of boundary conditions on K namely

$K(0)=0=K(1)$, we get

$$\int_0^1 \left|\left(D^2-a^2\right)K\right|^2 dz + 2p_2\sigma_r\int_0^1 \left(|DK|^2+a^2|K|^2\right)dz$$
$$+p_2^2|\sigma|^2\int_0^1 |K|^2 dz = \int_0^1 |DW|^2 dz \quad (34)$$

Since $p_2\rangle 0$ and $\sigma_r \ge 0$, therefore the equation (34) give,

$$\int_0^1 \left|\left(D^2-a^2\right)K\right|^2 dz \langle \int_0^1 |DW|^2 dz \quad (35)$$

And

$$\int_0^1 |K|^2 dz \langle \frac{1}{p_2^2|\sigma|^2}\int_0^1 |DW|^2 dz \quad , \quad (36)$$

It is easily seen upon using the boundary conditions (23) that

$$\int_0^1 \left(|DK|^2+a^2|K|^2\right)dz =$$ Real part of $$\left\{-\int_0^1 K^*\left(D^2-a^2\right)Kdz\right\} \le \left|\int_0^1 K^*\left(D^2-a^2\right)Kdz\right| ,$$

$$\le \int_0^1 \left| K^*\left(D^2 - a^2\right)K \right| dz \le \int_0^1 \left| K^* \right| \left| \left(D^2 - a^2\right)K \right| dz ,$$

$$= \int_0^1 |K| \left| \left(D^2 - a^2\right)K \right| dz$$

$$\le \left\{ \int_0^1 |K|^2 dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 \left| \left(D^2 - a^2\right)K \right|^2 dz \right\}^{\frac{1}{2}} ,$$

(Utilizing Cauchy-Schwartz-inequality)

Upon utilizing the inequality (35) and (36), we get

$$\int_0^1 \left(|DK|^2 + a^2 |K|^2 \right) dz \le \frac{1}{p_2 |\sigma|} \int_0^1 |DW|^2 dz , \qquad (37)$$

This completes the proof of lemma.
We prove the following theorem:

Theorem 1: If $R \rangle 0$, $R_s \rangle 0$, $F \rangle 0$, $Q \rangle 0$, $T_A \rangle 0$, $P_l \rangle 0$, $p_1 \rangle 0$, $p_2 \rangle 0$, $p_3 \rangle 0$, $\sigma_r \ge 0$ and $\sigma_i \ne 0$ then the necessary condition for the existence of non-trivial solution $(W, \Theta, \Gamma, K, Z, X)$ of equations (17) – (22), together with boundary conditions (23) is that

$$R_s \rangle \left(\frac{4\pi^4}{E' p_3} \right) \left\{ \left(\frac{1}{\varepsilon} + \frac{F}{P_l} \right) \left(1 - T_A P_l^2 \right) - \left(\frac{Q p_2}{\pi^2} \right) \right\} .$$

Proof: Multiplying equation (17) by W^* (the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{P_l}(1 + \sigma F) \right] \int_0^1 W^* \left(D^2 - a^2 \right) W dz = -Ra^2 \int_0^1 W^* \Theta dz$$
$$+ R_s a^2 \int_0^1 W^* \Gamma dz - T_A \int_0^1 W^* DZ dz + Q \int_0^1 W^* D\left(D^2 - a^2 \right) K dz , \qquad (38)$$

Taking complex conjugate on both sides of equation (21), we get

$$\left(D^2 - a^2 - E p_1 \sigma^* \right) \Theta^* = -W^* , \qquad (39)$$

Therefore, using (39), we get

$$\int_0^1 W^* \Theta dz = -\int_0^1 \Theta \left(D^2 - a^2 - E p_1 \sigma^* \right) \Theta^* dz , \qquad (40)$$

Taking complex conjugate on both sides of equation (22), we get

$$\left(D^2 - a^2 - E' p_3 \sigma^* \right) \Gamma^* = -W^* , \qquad (41)$$

Therefore, using (41), we get

$$\int_0^1 W^* \Gamma dz = -\int_0^1 \Gamma \left(D^2 - a^2 - E' p_3 \sigma^* \right) \Gamma^* dz , \qquad (42)$$

Also taking complex conjugate on both sides of equation (18), we get

$$\left[\frac{\sigma^*}{\varepsilon} + \frac{1}{P_l}(1 + \sigma^* F) \right] Z^* - Q D X^* = D W^* , \qquad (43)$$

Therefore, using (43), we get

$$\int_0^1 W^* DZ dz = -\int_0^1 DW^* Z dz = -\left[\frac{\sigma^*}{\varepsilon} + \frac{1}{P_l}(1 + \sigma^* F) \right] \int_0^1 Z^* Z dz + Q \int_0^1 Z D X^* dz , \qquad (44)$$

Integrating by parts the third term on left hand side of equation (44) and using equation (20), and appropriate boundary condition (23), we get

$$\int_0^1 W^* DZ dz = -\left[\frac{\sigma^*}{\varepsilon} + \frac{1}{P_l}(1 + \sigma^* F) \right] \int_0^1 Z^* Z dz + Q \int_0^1 X \left(D^2 - a^2 - p_2 \sigma \right) X^* dz , \qquad (45)$$

Also taking complex conjugate on both sides of equation (19), we get

$$\left[D^2 - a^2 - p_2 \sigma^* \right] K^* = -D W^* , \qquad (46)$$

Therefore, utilizing equation (45), and appropriate boundary condition (23), we get

$$\int_0^1 W^* D\left(D^2 - a^2 \right) K dz = -\int_0^1 DW^* \left(D^2 - a^2 \right) K dz = \int_0^1 K \left(D^2 - a^2 \right) \left(D^2 - a^2 - p_2 \sigma^* \right) K^* dz , \qquad (47)$$

Substituting (40), (42), (45) and (47), in the right hand side of equation (38), we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1 W^*\left(D^2-a^2\right)Wdz = Ra^2\int_0^1 \Theta\left(D^2-a^2-Ep_1\sigma^*\right)\Theta^* dz - R_s a^2\int_0^1 \Gamma^*\left(D^2-a^2-E' p_3\sigma^*\right)\Gamma dz$$

$$+T_A\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 Z^* Zdz - T_A Q\int_0^1 X\left(D^2-a^2-p_2\sigma\right)X^* dz$$

$$+Q\int_0^1 K^*\left(D^2-a^2\right)^2 Kdz - Qp_2\sigma^*\int_0^1 K^*\left(D^2-a^2\right)Kdz \quad , (48)$$

Integrating the terms on both sides of equation (48) for an appropriate number of times and making use of the appropriate boundary conditions (23), we get

$$\left[\frac{\sigma}{\varepsilon}+\frac{1}{P_l}(1+\sigma F)\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz = Ra^2\int_0^1\left(|D\Theta|^2+a^2|\Theta|^2+Ep_1\sigma^*|\Theta|^2\right)dz$$

$$-R_s a^2\int_0^1\left(|D\Gamma|^2+a^2|\Gamma|^2+E' p_3\sigma^*|\Gamma|^2\right)dz - T_A\left[\frac{\sigma^*}{\varepsilon}+\frac{1}{P_l}(1+\sigma^* F)\right]\int_0^1 |Z|^2 dz - T_A Q\int_0^1\left(|DX|^2+a^2|X|^2+p_2\sigma|X|^2\right)dz$$

$$-Q\int_0^1\left(\left|D^2K\right|^2+2a^2|DK|^2+a^4|K|^2\right)dz - Qp_2\sigma^*\int_0^1\left(|DK|^2+a^2|K|^2\right)dz \quad , (49)$$

Now equating imaginary parts on both sides of equation (49), and cancelling $\sigma_i(\neq 0)$, we get

$$\left[\frac{1}{\varepsilon}+\frac{F}{P_l}\right]\int_0^1\left(|DW|^2+a^2|W|^2\right)dz = -Ra^2Ep_1\int_0^1 [\Theta[^2 dz + R_s a^2 E' p_3\int_0^1 |\Gamma|^2 dz$$

$$+T_A\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)\int_0^1 [Z[^2 dz - T_A Qp_2\int_0^1 |X|^2 dz + Qp_2\int_0^1\left(|DK|^2+a^2|K|^2\right)dz \quad ,(50)$$

Now R 〉 0, $R_s\rangle 0$, $Q\rangle 0$, $\varepsilon\rangle 0$ and $T_A\rangle$ 0, utilizing the inequalities (25) ,(33) and (37), the equation (50) gives,

$$\left[\left\{\left(\frac{P_l+\varepsilon F}{\varepsilon P_l}\right)-\frac{T_A}{|\sigma|^2}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)-\frac{Q}{|\sigma|}\right\}\right]$$

$$\int_0^1 |DW|^2 dz + a^2\left[\left(\frac{1}{\varepsilon}+\frac{F}{P_l}\right)-\frac{R_s}{E' p_3|\sigma|^2}\right] \quad (51)$$

$$\int_0^1 |W|^2 dz + Ra^2Ep_1\int_0^1 |\Theta|^2 dz \langle 0$$

Therefore, we must have

$$|\sigma|^2\langle Maximumof\left[\begin{matrix}\left\{\frac{1}{4}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2\left(Q+\sqrt{Q^2+4T_A}\right)^2\right\}\\ ,\left\{\frac{R_s}{E' p_3}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\right\}\end{matrix}\right] \quad (52)$$

Hence, if $\sigma_r \geq 0$ and $\sigma_i \neq 0$, then

$$|\sigma|^2\langle Maximumof\left[\begin{matrix}\left\{\frac{1}{4}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2\left(Q+\sqrt{Q^2+4T_A}\right)^2\right\}\\ ,\left\{\frac{R_s}{E' p_3}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\right\}\end{matrix}\right], \quad (53)$$

And this completes the proof of the theorem.

Theorem 2: If R〈 0, $R_s\langle 0$, F〉 0, $P_l\rangle 0$, $p_1\rangle 0$, $p_3\rangle 0$, $\sigma_r \geq 0$ and $\sigma_i \neq 0$ then the necessary condition for the existence of non-trivial solution (W,Θ,Z,K,X,Γ) of equations (17) – (20), together with boundary conditions (21) is that

$$|\sigma|^2\langle Maximumof\left[\begin{matrix}\left\{\frac{1}{4}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)^2\left(Q+\sqrt{Q^2+4T_A}\right)^2\right\}\\ ,\left\{\frac{|R|}{Ep_1}\left(\frac{\varepsilon P_l}{P_l+\varepsilon F}\right)\right\}\end{matrix}\right] \quad (54)$$

Proof: Replacing R and R_s by $-|R|$ and $-|R_s|$, respectively in equations (17) – (22) and proceeding exactly as in Theorem 1 and utilizing the inequality (27), we get the desired result.

CONCLUSIONS

The inequality (53) for $\sigma_r \geq 0$ and $\sigma_i \neq 0$, can be written as

$$\sigma_r^{\ 2} + \sigma_i^{\ 2} \langle Maximum of \left[\left\{ \frac{1}{4}\left(\frac{\varepsilon P_l}{P_l + \varepsilon F} \right)^2 \left(Q + \sqrt{Q^2 + 4T_A} \right)^2 \right\} , \left\{ \frac{R_s}{E^{'} p_3} \left(\frac{\varepsilon P_l}{P_l + \varepsilon F} \right) \right\} \right]$$

The essential content of the theorem, from the point of view of linear stability theory is that for the thermosolutal Veronis (1965) type configuration of Rivlin-Ericksen viscoelastic fluid in the presence of uniform vertical rotation in a porous medium, having top and bottom bounding surfaces of infinite horizontal extension, with any arbitrary combination of dynamically free and rigid boundaries in a porous medium, the complex growth rate of an arbitrary oscillatory motions of growing amplitude, lies inside a semi-circle in the right half of the $\sigma_r \sigma_i$ - plane whose centre is at the origin and radius is equal to

$$Maximum of \left[\left\{ \frac{1}{4}\left(\frac{\varepsilon P_l}{P_l + \varepsilon F} \right)^2 \left(Q + \sqrt{Q^2 + 4T_A} \right)^2 \right\} , \left\{ \frac{R_s}{E^{'} p_3} \left(\frac{\varepsilon P_l}{P_l + \varepsilon F} \right) \right\} \right]$$

where R_s is the thermosolutal Rayleigh number, Q is the Chandrasekhar number, T_A is the Taylor number, F is the viscoelasticity parameter, p_3 is the thermosolutal prandtl number, ε is the porosity and P_l is the medium permeability. The result is important since it hold for any arbitrary combinations of dynamically free and rigid boundaries. The similar conclusions are drawn for the thermosolutal configuration of Stern (1960) type of Rivlin-Ericksen viscoelastic fluid of infinite horizontal extension in the presence of uniform vertical magnetic field in a porous medium, for any arbitrary combination of free and rigid boundaries at the top and bottom of the fluid from Theorem 2.

REFERENCES

[1] Chandrasekhar, S. Hydrodynamic and Hydromagnetic Stability, 1981, Dover Publication, New York.

[2] Veronis, G., On finite amplitude instability in the thermohaline convection, J. Marine. Res., 23(1965), pp.1-17.

[3] Stern, M. E., The salt fountain and thermohaline convection, Tellus 12 (1960), 172-175.

[4] Bhatia, P.K. and Steiner, J.M., Convective instability in a rotating viscoelastic fluid layer, Zeitschrift fur Angewandte Mathematik and Mechanik 52 (1972), 321-327.

[5] Sharma, R.C., Effect of rotation on thermal instability of a viscoelastic fluid, Acta Physica Hungarica 40 (1976), 11-17.

[6] Oldroyd, J.G., Non-Newtonian effects in steady motion of some idealized elastic-viscous liquids, Proceedings of the Royal Society of London A245 (1958), 278-297.

[7] Rivlin, R.S. and Ericksen, J.L., Stress deformation relations for isotropic materials, J. Rat. Mech. Anal. 4 (1955), 323.

[8] Sharma, R.C. and Kumar, P., Effect of rotation on thermal instability in Rivlin-Ericksen elastic-viscous fluid, Zeitschrift fur Naturforschung 51a (1996), 821-824.

[9] Kumar, P., Mohan, H. and Lal, R., Effect of magnetic field on thermal instability of a rotating Rivlin-Ericksen viscoelastic fluid, Int. J. of Maths. Math. Scs., Vol-2006 article ID 28042, pp. 1-10.

[10] Nield D. A. and Bejan, A., Convection in porous medium, springer, 1992.

[11] Pellow, A., and Southwell, R.V., On the maintained convective motion in a fluid heated from below. Proc. Roy. Soc. London A, 1940, 176, 312-43.

[12] Banerjee, M.B., Katoch, D.C., Dube,G.S. and Banerjee, K., Bounds for growth rate of perturbation in thermohaline convection. Proc. R. Soc. A, 1981,378, 301-04

[13] Banerjee, M. B., and Banerjee, B., A characterization of non-oscillatory motions in magnetohydronamics. Ind. J. Pure & Appl Maths., 1984, 15(4): 377-382

[14] Gupta, J.R., Sood, S.K., and Bhardwaj, U.D., On the characterization of nonoscillatory motions in rotatory hydromagnetic thermohaline convection, Indian J. pure appl.Math., 1986, 17(1), pp 100-107.

[15] Banyal, AS (2012). A characterization of Rivlin-Ericksen viscoelastic fluid in the presence of rotation, International Journal of Physics and Mathematical sciences, 2(2) 58-64.

[16] Sharma, R.C., Sunil and Pal, M., Thermosolutal convection in Rivlin-Ericksen rotating fluid in porous medium in hydromagnetics, indian J. pure appl. Math., 32(1) (2001), pp. 143-156.